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A Differential Method of Optical Calculation

Part I: Variables

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Abstract

In an optical system which has rotational symmetry about the optical axis, it is generally possible to define the incident ray with three proper quantities and to classify various aberration terms with them.

By adopting fundamental variables as such quantities the image formation of an optical system is discussed theoretically and formulas for calculation are derived.

Then subsidiary variables that are suited to express the geometrical relation between a ray and the optical axis are defined to be used for later calculations as auxiliary quantities especially suitable for differential calculations.

Further by introducing principal variables which are used to fix the position of a point on the ray, the object and image points adopted in this treatise are defined, and the quantities corresponding to the magnification and the focal length for the paraxial ray are given.

It is shown that principal variables in every space are connected by linear relations and therefore can be expressed in differential forms of the 1st order. To make use of these relations, a standard optical system in which rays are fixed by taking an object point at infinity and another point at the entrance pupil is introduced.

A general optical system in which an object point is located at a finite distance can be transformed to a standard optical system by placing an ideal optical system in front of it, and as a result the treatment becomes unified.

In the standard optical system, subsidiary variables are more convenient to specify the incident ray and to classify aberration terms than fundamental variables.

Besides, formulas expressing relations among the above mentioned variables are derived and used extensively in later calculation.

1. Introduction

Basic properties desirable for variables used in the following calculations are considered. A Cartesian co-ordinate system x, y, z of which the origin is any point on the optical axis and the x-axis coincide with the optical axis is adopted.

With these co-ordinates two points on the ray are expressed as $Q(x, y, z)$ and $Q_*(x_*, y_*, z_*)$.

Hereafter, these coordinates and all the other quantities concerning the length are, if not mentioned, expressed in terms of optical path lengths, that is the length multiplied by the refractive index.

Let us call the following three quantities that are given by the co-ordinates of Q and Q_* as the fundamental variables

$$a=y^2+z^2,\quad b=yy_*+zz_*,\quad c=y_*^2+z_*^2 \tag{I-1}$$

In an optical system that has a rotational symmetry about the optical axis, the aberration terms are classified according to these three independent variables in the object space[1]. In other words, the quantity X' that is suited to describe the condition of the image can be expanded in terms of a, b and c with the use of the differential coefficients of X' by them, as

$$X'=X'_0+(X'_a a+X'_b b+X'_c c)$$
$$+(\tfrac{1}{2}X'_{aa}a^2+X'_{ab}ab+\cdots\cdots)+\cdots\cdots \tag{I-2}$$

If we differentiate X', for example, by a and put $a=b=c=0$, then $dX'/da=X'_a$. That is to say the aberration coefficient which is equivalent to the expantion coefficient of X' can be obtained with such differentiation.

If variables such as y, z and so on are taken, the coefficient such as X'_a can not be obtained with differentiation of the 1st order but with that of the 2nd order.[2][3]

Therefore such quantities as fundamental variables defined in (I-1) are convenient to discuss the aberration of an optical system and are being used mainly in the aberration theory.

To perform the ray tracing and the differential calculation in an unifiied way, three subsidiary variables closely related to the fundamental variables and more convenient for these calculations are introduced in this treatise.

But a point on a ray can not be determined completely with three variables alone. For example, x and x_* are inditerminate even when the fundamental variables are given.

Hence in this treatise principal variables that are suited to represent a point on the ray are employed.

Subsidiary and principal variables will be explained in the following two sections.

---

1) Masazo Kiuti: *Kogaku* (Iwanami Butsurigaku Koza, Iwanami, Tokyo, 1941).
2) A. E. Conrady: *Applied Optics and Optical Design* (Oxford Univ. press, London, 1929).
3) D. P. Feder: J. Opt. Soc. Amer. **41** (1951) 630.

2. Subsidiary Variables

To express the position and direction of rays, a rectangular co-ordinate system ξ, η, ζ is used also.

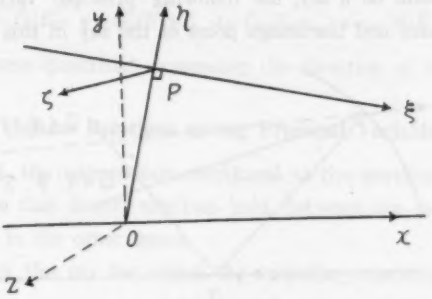

Fig. I-1

The origin P of ξ, η, ζ co-ordinates is the point of intersection between a ray under consideration which is taken as ξ-axis and the perpendicular from O to ξ-axis which is taken as η-axis.

Let h be the length of $\overrightarrow{OP}$ and let the direction cosines of the axes ξ, η, ζ with respect to the co-ordinate system x, y, z be (g_{11}, g_{12}, g_{13}), (g_{21}, g_{22}, g_{23}), (g_{31}, g_{32}, g_{33}) respectively, and define A, B, C, D and H by the following expressions with the use of these quantities

$$A=g_{11},\quad B=-hg_{21},\quad C=hg_{31},\quad H=h^2,\quad D=1-A^2. \tag{I-3}$$

As is seen from these expressions, both A and D represent the relation between the direction of the ray and x-axis. $-B$ which is hg_{21}, is the x component of $\overrightarrow{OP}$ in Fig. I-1. C is a variable peculiar to the skew ray and is equal to zero when $g_{31}=0$.

It will be shown later that C becomes invariant throughout the optical system. H is a variable which depends only on the length.

We call these the subsidiary variables, by which the condition of a ray in relation to the optical axis is determined. There remains however, a freedom of rotation of the ray about the optical axis. Hence the condition with respect to y- and z- directions will be treated in the next section.

H, B and D are small quantities of the 2nd order and are generally taken as three independent variables corresponding to the fundamental variables a, b and c.

The following relation holds from (I-3) and will be used for check of

calculation.

$$HD=B^2+C^2. \tag{I-4}$$

3. Principal Variables

To express a point on a ray, the following principal variables are used with which the object point and the image point of the ray in this theory are defined.

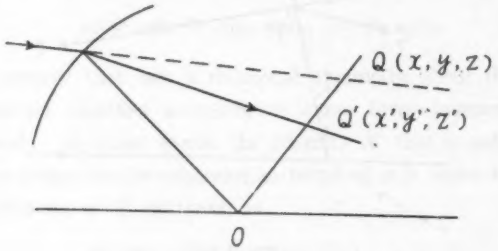

Fig. I-2

Consider a ray before and after passing a spherical surface in Fig. I-2. Let $Q(x, y, z)$ be a point on the incident ray and $Q'(x', y', z')$ be the point of the intersection between the refracted ray and $\overrightarrow{OQ}$.

If we put $E/E'=\overrightarrow{OQ'}/\overrightarrow{OQ}$, we obtain $E/E'=y'/y=z'/z$. This designation of Q, Q' and E, E' is extended to the whole optical system.

The plane which contains x-axis and OQ (and therefore $\overrightarrow{OQ'}$) is invariant throughout the optical system. We call this plane the fundamental plane. Q and Q' are the intersections between the ray and this fundamental plane. By these points which are the generallized so-called diapoints of Herzbergar[4], the object and image points of this surface are defined. In the case in which the ray is always in one plane, such as meridional plane, they correspond to the image points of the sagittal rays.

Using these relations, we express any point on the ray with the principal variables E, G or F and the principal variable in object space E_0, as

$$x=-G/E,\quad y=E_0/E\cdot y_0,\quad z=E_0/E\cdot z_0,\quad s=-F/E. \tag{I-5}$$

The suffix o signifies the quantity in object space hereafter.

In the object space, (I-5) becomes by taking E_0 and G_0 instead of E and G respectively as

$$x=-G_0/E_0,\quad y=y_0,\quad z=z_0,\quad s=-F_0/E_0. \tag{I-6}$$

4) M. Helzberger: *Modern Geometrical Optics* (Interscience Publishers, Inc., New York, 1958).

If the intersection between the object plane and the optical axis is taken as the origin, x and therefore G_0 become zero and only E_0 remains.

When the object plane is at infinity, E_0 becomes to zero and only G_0 remains as seen from (I-6). In this case the following expressions containing G_0 instead of E_0 are used.

$$x=-G/E,\quad y=G_0/E\cdot y_0,\quad z=G_0/E\cdot z_0,\quad s=-F/E \tag{I-7}$$

where y_0 and z_0 become quantities expressing the direction of the ray.

4. Linear Relations among Principal Variables

On the basis of the expressions mentioned in the previous section, it will be shown in this section that linear relations hold between the principal variables in one space and those in the other space.

Here a point on the ray for which the subsidiary variables are given is considered.

The principal variables by themselves have no physical meanings and only the ratios of one to another can express practical quantities as seen from (I-5) and (I-7). In object space G_0/E_0 in (I-6) corresponds to this quantity.

The expressions of (I-5) are again considered. We may express x as $x=-\dfrac{G}{E_0}\Big/\dfrac{E}{E_0}$ where E/E_0 and x are unaltered when the ratio G_0/E_0 is a constant, because in this case, from (I-6), x, y, z and therefore the point remains unchanged. Accordingly, G/E_0 similarly to E/E_0 remains unaltered. This relation holds only when E and G are both expressed by linear combinations of E_0 and G_0.

These linearlities of principal variables make our calculation rational and especially the differential calculation simple. It is mainly because of this linearity that the principal variables are employed in our calculation.

5. A Standard Optical System

In this section, by assuming the linear relations which hold among principal variables as shown in the previous section, an optical system named the standard optical system is derived.

It means, for example, that the values of any two points on a ray determines the value of any other point on the same ray by these linear relations. Take the center of the entrance pupil as the origin and consider the two points

$$G_0=1,\quad E_0=0$$

and

$$G_0=0,\quad E_0=1 \tag{I-8}$$

on the incident ray. Of these, one is the object point at infinity and the other is the point on the entrance pupil. We call such an optical system involving these two points the standard optical system.

If X is the principal variable in any space, the value of X can then be represented by X_G and X_E, the derivatives of X by E_0 and G_0 because of the linearity (Hereafter, the suffices, if not otherwise mentioned, signify quantities in the object space). As regards any point in general with principal variables E_0 and G_0 in object space, the value of X can easily be obtained as

$$X=X_E E_0+X_G G_0. \tag{I-9}$$

It will be clear that the differential relations among the principal variables are none other than what is described above, and that differential coefficients of the order higher the 2nd do not exist.

In object space, taking (I-9) for $X=G$ and $X=E$ and using the values of E_0 and G_0 in (I-8), we obtain the evidient expressions

$$G_G=1, \quad G_E=0$$

and

$$E_G=0, \quad E_E=1. \tag{I-10}$$

Let us employ such a differential expression of principal variables in every case hereafter.

Thus (I-5) and (I-7) are replaced by the following expressions

$$\begin{aligned} &E_G x=-G_G, \quad E_G y=y_0, \quad E_G z=z_0, \quad E_G s=-F_G \\ &E_E x_*=-G_E, \quad E_E y_*=y_{*0}, \quad E_E z_*=z_{*0}, \quad E_E s_*=-F_E \end{aligned} \tag{I-11}$$

where suffix $*$ represents a quantity referred to the pupil.

Then from (I-1) and (I-11) the expression of fundamental variables is

$$E^2_G a=a_0, \quad E_G E_E b=b_0, \quad E^2_E c=c_0. \tag{I-12}$$

6. Incident Rays and Subsidiary Variables

As stated above, the object plane and the entrance pupil can be determined by principal variables. We now consider how the position of an object point and the direction of a ray are expressed by subsidiary variables.

Take a standard optical system in which the object point at infinity is on the meridional plane and the center of the entrance pupil is the origin O, and investigate the relations between the rays from the object point and subsidiary variables.

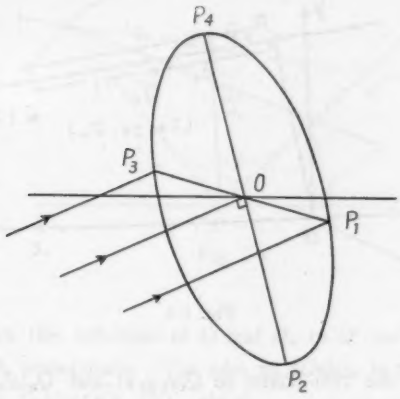

Fig. I-3

In Fig. I-3, p_1, p_2, p_3 and p_4 are points equally distant from O and are on the plane which includes O and is perpendicular to the principal ray. From the expression (I-3) it will be clear that

1 Since the rays from the object point are parallel to one another, g_{11} and therefore D_0 are both constants.

2 As regards the meridional rays onto $\overline{P_2P_4}$, g_{31} and therefore C are both equal to zero.

3 As regards the sagittal rays onto $\overline{P_1P_3}$, g_{21} and therefore B_0 are both equal to zero.

4 As regards the zonal ray onto $\overset{\frown}{P_1P_2P_3P_4}$, H is constant.

As shown above, the subsidiary variables in object space, namely H_0, B_0, D_0 or C, are the variables that suit to specify the incident ray of the standard optical system and consequently to classify the aberration terms in this case.

But they are insufficient to describe the incident ray simply in a general optical system. Use of other variables such as the fundamental variables a, b and c is also effective as is seen from (I-1). Nevertheless, we shall find the way how a general optical system can be transformed into a standard optical system in Section I-9.

7. Fundamental Relations

Fundamental relations between the quantities referred to two points on a ray are investigated.

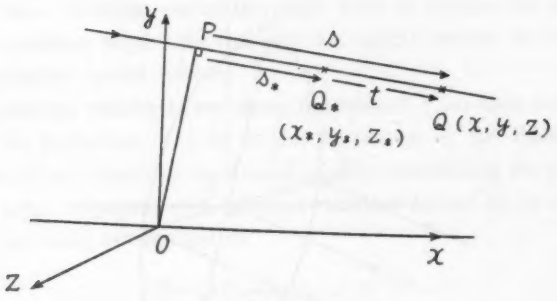

Fig. I-4

In Fig. I-4, let the two points be $Q(x, y, z)$ and $Q_*(x_*, y_*, z_*)$. Put $\overrightarrow{PQ}=s$, $\overrightarrow{PQ}_*=s_*$ and $\overrightarrow{Q_*Q}=t$. Then there exist the following relations between these quantities and fundamental variables a, b and c.

$$\begin{aligned} ts&=(x^2+a)-(xx_*+b)\\ ts_*&=(xx_*+b)-(x_*^2+c)\\ t^2&=(x^2+a)-2(xx_*+b)+(x_*^2+c) \end{aligned} \tag{I-13}$$

With the use of these quantities, the quantities regarding h and the direction cosines are expressed as

$$\begin{aligned} tg_{11}&=x-x_*, \quad thg_{21}=sx_*-s_*x\\ tg_{12}&=y-y_*, \quad thg_{32}=sy_*-s_*y\\ tg_{13}&=z-z_*, \quad thg_{23}=sz_*-s_*z\\ thg_{31}&=yz_*-y_*z\\ thg_{32}&=zx_*-z_*x\\ thg_{33}&=xy_*-x_*y\\ t^2h^2&=(x^2+a)(x_*^2+c)-(xx_*+b)^2. \end{aligned} \tag{I-14}$$

From (I-14) and (I-3), we have $t(g_{12}z-g_{13}y)=yz_*-y_*z=thg_{31}=Ct$, namely

$$C=g_{12}z-g_{13}y. \tag{I-15}$$

Combining this relation with (I-14) and using (I-11) we obtain

$$\begin{aligned} CG_0&=h(g_{32}y_0+g_{33}z_0)\\ CF_0&=h(g_{22}y_0-g_{23}z_0). \end{aligned} \tag{I-16}$$

Further a fundamental relation, in other words a quantity invariant throughout the optical system, is derived.

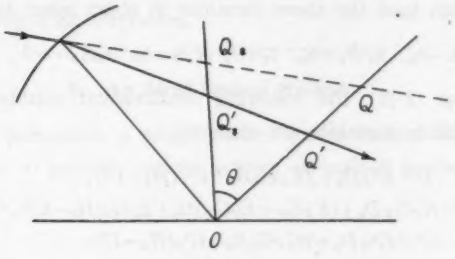

Fig. I-5

Fig. I-5 shows the relations of Q and Q_* to Q' and Q'_* that are the image points of Q and Q_* respectively. The area of ΔOQQ_* is $t \cdot h = \overrightarrow{OQ}_*$. $\overrightarrow{OQ}$ sin θ and that of $\Delta OQ'Q'_*$ is $t' \cdot h' = \overrightarrow{OQ'}$. $\overrightarrow{OQ'}_*$ sin θ.

Taking the ratio between the two and assuming the relation $h=h'$ that will be explained later, we have $E'_G E'_E t' = E_G E_E t$.

If this treatment is extended to the object space

$$E_G E_E t = t_0 \tag{I-17}$$

is obtained, where t_0 represents a quantity invariant and its value will be determined in the following section.

8. Relation among Variables

On the basis of the relations derived in the previous section, various fundamental relations among principal and subsidiary variables will be obtained.

First, multiplying the first formula of (I-14) by $E_G E_E$, we have $-t_0 A = G_G E_E - G_E E_G$. From (I-10) it becomes in object space, as

$$t_0 A_0 = -1 . \tag{I-18}$$

Therefore, the fundamental relation

$$G_G E_E - G_E E_G = A/A_0 \tag{I-19}$$

is obtained. Similarly, by multiplying (I-13) by $E_E^2 E_G^2$ and using the expressions of (I-1) and (I-14), we obtain

$$\begin{aligned} t^3_0 D &= E^2_E a_0 - 2E_E E_G b_0 + E^2_G c_0 \\ -t^2_0 B &= G_E E_E a_0 - (G_G E_E + G_E E_G) b_0 + G_G E_G c_0 \\ t^2_0 (H - C^2) &= G^2_E a_0 - 2G_G G_E b_0 + G^2_G c_0 \\ t^3_0 C^2 &= a_0 c_0 - b^3_0 . \end{aligned} \tag{I-20}$$

If (I-10) is again used the above formulas in object space are reduced to

$$t_0^2 D_0 = a_0, \quad t_0^2 B_0 = b_0, \quad t_0^2 (H_0 - C^2) = c_0, \quad t_0 A_0 = -1. \tag{I-21}$$

From (I-20) and (I-21), the following fundamental relation formulas among principal and subsidiary variables are obtained

$$\begin{aligned} D &= E_E^2 D_0 + 2E_E E_G B_0 + E_G^2 (H_0 - C^2), \\ B &= E_E G_E D_0 + (E_E G_G + E_G G_E) B_0 + E_G G_G (H_0 - C^2), \\ H - C^2 &= G_E^2 D_0 + 2G_E G_G B_0 + G_G^2 (H_0 - C^2). \end{aligned} \tag{I-22}$$

It is seen that subsidiary variables can be computed from principal variables in any space. This corresponds to the fact that, when the positions of any two points on a ray are given, the relation between the ray and the optical axis is determined.

Therefore, the complete expression of the emerging ray in image space is possible with four principal variables G_G, E_G, G_E and E_E. Hence we can take these principal variables as the main quantities in calculation hereafter.

Also (I-22) makes the differential calculation of subsidiary variables simple, especially that of the higher order, because of the linear relations among subsidiary variables D, B and H.

Besides, relations which are to be useful later on, are here derived. The combination of (I-16) with (1-3) gives

$$G_G = F_G A + E_G B, \quad F_G^2 + E_G^2 H = G_G^2 + D/A_0^2. \tag{I-23}$$

These relations are convenient for checking of calculation. Modifiing (I-23), we have $F_G A = G_G (A^2 + D) - E_G B$, then D_G is defined as

$$P_G = F_G - G_G A = \frac{1}{A}(G_G D - E_G B). \tag{I-24}$$

For an axial ray, $B_0 = D_0 = 0$ and therefore $P = 0$, hence

$$F_G = G_G A \tag{I-25}$$

is given.

Further, from (I-22) and (-24), $G_G D - E_G B = (G_G E_E - E_G G_E)(E_G B_0 + E_E D_0) = \frac{A}{A_0}(E_G B_0 + E_E D_0)$ is derived, then P_G becomes

$$A_0 P_G = E_G B_0 + E_E D_0. \tag{I-26}$$

If E_0 is used instead of G_0, P_E is given by

$$-A_0 P_E = E_E B_0 + E_G (H_0 - C^2). \tag{I-27}$$

These relations (I-26) and (I-27) are used later for ray tracing calculation and

for calculation of wave front aberration.

9. An Ideal Optical System

To simplify our calculation, a proper optical system, named the standard optical system, with which a general optical system is unified to the standard optical system, is introduced.

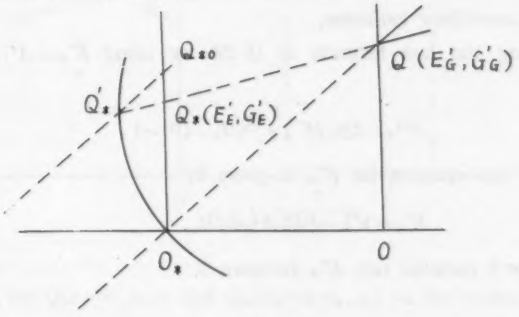

Fig. I-6

In Fig. I-6, let the intersection between the object plane OQ and the optical axis be the origin O. Let Q be the object point and Q_* be the centre of the entrance pupil. Placing an adequate optical system at Q_* on the optical axis with $\overrightarrow{Q_*O}$ as its focal length f, we can fix the following relations between E_0, G_0 and E', G', the principal variables in front and behind the optical system,

$$\begin{aligned} &G'_G=1, \quad E'_G=-J \\ &G'_E=0, \quad E'_E=A'/A_0 \end{aligned} \tag{I-28}$$

where $J=1/f$. These expressions satisfy the relation $G'_G E'_E-G'_E E'_G=A^1/A_0$ of (I-19), and mean that G'_G and E'_G are constants and the rays from a point at infinity ($E_0=0$, $G_0=1$) concentrate to a point Q through this optical system as an aberration-free image, and that a point ($E_0=1$, $G_0=0$) on the entrance pupil forms its image on the same plane because of $G'_E=0$.

Also it is seen that a point $Q_*(E_0=1, JG_0=A'/A_0-1)$ on the spherical surface that passes through Q_* and has its centre at Q remains unchanged because of $G'=G_0$ and $E'E_0$ as derived from (I-9).

We shall call such an optical system the ideal optical system. A general optical system can be transformed into a standard optical system by placing the ideal optical system in front of it, whereby our problem can be treated in this unified

way, and from this standpoint various calculations will be dealt with hereafter.

Besides, relations that are required in later sections are derived by using (I-22) as

$$H'=H_0,\quad B'=E'_E B_0-J(H_0-C^2)$$
$$D'=E'^2_E D_0-2JE'_E B_0+J^2(H_0-C^2). \tag{I-29}$$

Here $H'=H_0$, and it is obvious that the relations 1~4 stated in Section I-6 hold as they are. Hence, an incident ray in a general optical system can clearly be specified by subsidiary variables.

Transforming the last formula of (I-29) by using $E'_E=A'/A_0$ of (I-28), we obtain

$$E'^2_E-2B_0JE'_E+J^2(H_0-C^2)=1. \tag{I-30}$$

Solution of this equation for E'_E is given by

$$E'_E=\sqrt{1-J^2H_0A^2_0}+JB_0. \tag{I-31}$$

Of cause, for a paraxial ray, E'_E becomes 1.

10. Linearity of Aberration Coefficients

It is shown that aberration coefficients of ordinary optical system can be obtained from the aberration coefficients, which is of the same order, of the standard optical system with linear relations.

Let us use principal and fundamental variables in this discussion.

In the standard optical system, the principal variables of the image point are expressed as

$$X'_G=X'_{G0}+(X'_{Ga}a+X'_{Gb}b+X'_{Gc}c)$$
$$+(\tfrac{1}{2}X'_{Gaa}a^2+X'_{Gab}ab+\cdots)+\cdots \tag{I-32}$$

where X'_{G0} is the value of X'_G for a paraxial ray, therefore the differential coefficients of X'_G are regarded as the aberration coefficients of X'_G.

From the calculation about pupil, X'_E is obtained which is expressed as

$$X'_E=X'_{E0}+(X'_{Ea}a+X'_{Eb}b+X'_{Ec}c)$$
$$+(\tfrac{1}{2}X'_{Eaa}a^2+X'_{Eab}ab+\cdots)+\cdots. \tag{I-33}$$

In an ordinary optical system, of which the principal variables are G and E, $X'=X'_G G+X'_E E$, and the following expression is obtained from the above formulas

$$X'=X'_a+(X'_a a+X'_b b+X'_c c)$$
$$+(X'_{aa}a^2+X'_{ab}ab+X'_{bb}b^2+\cdots)+\cdots$$

where

$$X'_a=X'_{Ga}G+X'_{Ea}E,\ X'_b=X'_{Gb}G+X'_{Eb}E,\cdots\cdots$$
$$X'_{aa}=\tfrac{1}{2}(X'_{Gaa}G+X'_{Eaa}E),\ X'_{ab}=X'_{Gab}G+X'_{Eab}E,\cdots\cdots \quad \text{(I-34)}$$

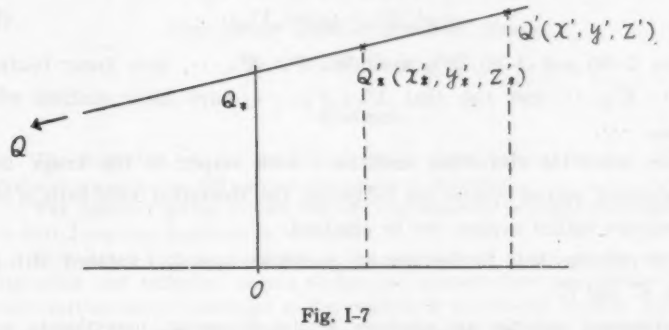

Fig. I-7

In Fig. I-7, let $Q(x=\infty, y, z)$ and $Q_*(x_*=0, y_*, z_*)$ be the object ponint and the point on the entrance pupil respectively in the standard optical system and let Q' (x', y', z') and $Q'_*(x'_*, y'_*, z'_*)$ be similar points in the ordinary optical system. then there are the relations

$$(x'-x'_*)y=y'-y'_*,\qquad (x'-x'_*)z=z'-z'_*$$
$$(x'-x'_*)y_*=x'y'_*-x'_*y',\ (x'-x'_*)y_*=x'z'_*-x'_*z'. \quad \text{(I-35)}$$

From these we have

$$(x'-x_*)^2a=a'-2b'+c'$$
$$(x'-x_*)^2b=-x'_*a'+(x'+x'_*)b'-x'c'$$
$$(x'-x_*)^2c=x'^2_*a'-2x'_*x'b'+x'^2c'. \quad \text{(I-36)}$$

If a', b' and c' are employed instead of a, b and c, (I-34) will be

$$X'=X'_0+(X'_{a'}a'+X'_{b'}b'+X'_{c'}c')$$
$$+(X'_{a'a'}a'^2+X'_{a'b'}a'b'+\cdots\cdots)+\cdots\cdots \quad \text{(I-37)}$$

in which $X'_{a'}\ X'_{b'},\cdots, X'_{a'a'}, X'_{a'b'},\cdots\cdots$are obtained by transforming (I-34) with the use of (I-36) as follows

$$(x'-x'_*)^2X'_{a'}=X'_a-x'^*X'_b+x'^2_*X'_c$$
$$(x'-x'_*)^2X'_{b'}=-2X'_a+(x'+x'_*)X'_b-2x'x'_*X'_c$$
$$(x'-x'_*)^2X'_{c'}=X'_a-x'X'_b+x'^2X'_c$$

and

$$(x'-x'_*)^4X'_{a'^2}=X'_{a^2}-x'_*X'_{ab}+x'^2_*(X'_{b^2}+X'_{ac})-x'^3_*X'_{bc}+x'^4_*X'_{c^2}$$

$$(x'-x'_*)^4X'_{a'b'}=-4X'_{a^2}+(x'+3x'_*)X'_{ab}-2x'_*(x'+x'_*)(X'_{b^2}+X'_{ac})+x'^2_*(3x'+x'_*)X'_{bc}-4x'x'^3_*X'_{c^2}\cdots\cdots. \qquad \text{(I-38)}$$

From (I-34) and (I-38), it is seen that $X'_{a'}, X'_{b'}, \cdots$, have linear relations with $X'_{Ga}, \cdots, X'_{Ea}, \cdots$and also that $X'_{a'^2}, X'_{a'b'}, \cdots$have linear relations with $X'_{Ga^2}, \cdots, X'_{Ea^2}, \cdots$.

Hence, when the aberration coefficients with respect to the image and pupil in the standard optical system are computed, the aberration coefficients of the image in an ordinary optical system can be obtained.

These relations hold further among quantities linearly combined with principal variables E' and G'.

Fundamental variables are employed in this discussion, nevertheless, subsidiary variables can be taken as such quantities by the simple reason that fundamental variables are more suitable to define incident rays in an ordinary optical system.

Result

Characteristic variables, namely the subsidiary and principal variables, are introduced, so that the ray tracing and differetial calculation will be made simple and easy.

Especially, it is noted that the linearity among principal variables will simplify the differential calculation hereafter.

Besides, by employing more variables than what are necessary to define the geometrical condition of a ray and the image point, several checking formulas and fundamental relations can be derived.

A Differential Method of Optical Calculation

Part II: Ray Tracing

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Abstract

The derivation of ray tracing formulas which involve the case of skew ray and are suited to differential calculation is described.

The standard optical system and the expression of principal variables used in Part I are also employed in this part.

It is shown that the variation of the direction of a ray in the cases of refraction and reflection at one surface and transfer from one surface to the next surface can be expressed as the rotation of co-ordinate system, one axis of which is the ray. According to this fundamental operation, the ray tracing formulas as regards principal and subsidiary variables are derived.

Besides, formulas for an aspherical surface are obtained by extending the above method and formulas for a plane surface are also derived.

It is shown by the ray tracing formulas that the linear relations among principal variables which are verified in Part I hold throughout the optical system.

Simplified formulas for tracing typical rays such as paraxial, axial and extra axial rays, are derived.

Further, choosing the wave front aberration as a quantity suited to describe the condition of the image, we obtain a method to compute this aberration with the use of the results of ray tracing.

1. Introduction

In the cases of refraction and transfer, we begin with the relation between the variation of a ray and that of the co-ordinate system ξ, η, ζ on the ray.

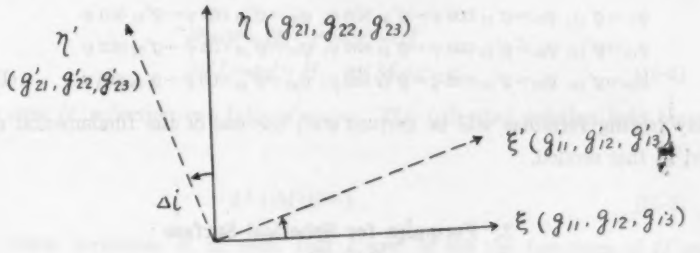

Fig. II-1

The ray before and after refraction is always in the incident plane, namely, the ξ–η plane, and ζ-axis remains unchanged.

Let the angle of rotation about ζ-axis be Δi, the sign of the rotation from ξ-axis to η-axis being taken as positive, then the relation between ξ, η and ξ, η becomes as shown in Fig. II-1 and the following relations hold among the x-, y-, z- components of these co-ordinates.

$$g'_{11}=g_{11}\cos\Delta i+g_{21}\sin\Delta i,\ g'_{21}=g_{21}\cos\Delta i-g_{11}\sin\Delta i,\ g'_{31}=g_{31}$$
$$g'_{12}=g_{12}\cos\Delta i+g_{22}\sin\Delta i,\ g'_{22}=g_{22}\cos\Delta i-g_{12}\sin\Delta i,\ g'_{32}=g_{32}$$
$$g'_{13}=g_{13}\cos\Delta i+g_{23}\sin\Delta i,\ g'_{23}=g_{23}\cos\Delta i-g_{13}\sin\Delta i,\ g'_{33}=g_{33} \qquad \text{(II-1)}$$

Next, for transfer, the direction of a ray does not change and the variation of co-ordinates are expressed with the rotation of η-and ζ-axis about ξ-axis.

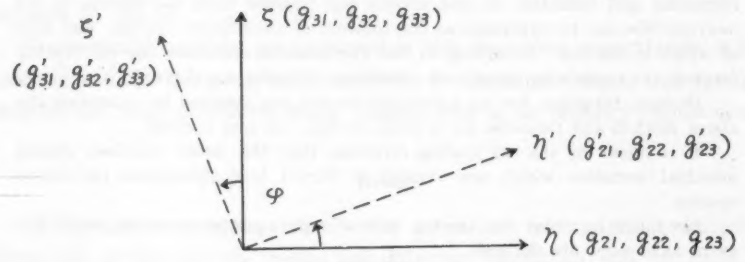

Fig. II-2

If φ is the angle of rotation, the sign of rotation from η-axis towards ζ-axis being taken as positive, the relations between η, ζ and η', ζ' are shown in Fig. II-2.

The relations among x-, y-, z- ,components of these co-ordinates are expressed as follows

$$g_{11}=g'_{11},\ g_{21}=g'_{21}\cos\varphi-g'_{31}\sin\varphi,\ g_{31}=g'_{31}\cos\varphi-g'_{21}\sin\varphi$$
$$g_{12}=g'_{12},\ g_{22}=g'_{22}\cos\varphi-g'_{32}\sin\varphi,\ g_{32}=g'_{32}\cos\varphi-g'_{22}\sin\varphi$$
$$g_{13}=g'_{13},\ g_{23}=g'_{23}\cos\varphi-g'_{33}\sin\varphi,\ g_{33}=g'_{33}\cos\varphi-g'_{23}\sin\varphi. \qquad \text{(II-2)}$$

Ray tracing formulas will be derived with the use of the fundamental relations derived in this section.

2. Formulas for Spherical Surfase

First, several equations concerning the refraction of a ray at a spherical surface are given below.

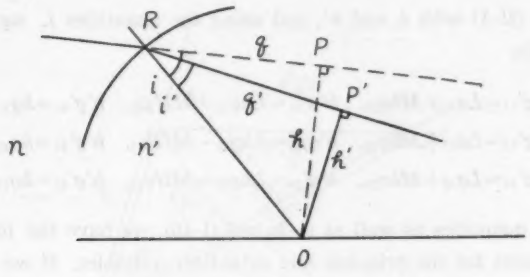

Fig. II-3

In Fig. II-3, let the intersection between the ray and the spherical surface be R, and put $\overrightarrow{RO}=r$. The sign of r is the same as that of x-component of $\overrightarrow{RO}$. Let the indices of refraction of spaces in front and behind the spherical surface be n and n' respectively, and put

$$p=nr, \quad p'=n'r \tag{II-3}$$

where r is a quantity which has no concern with refractive index. Let $\overrightarrow{OP}=h$, $OP'=h'$ and the angles between the normal at R and the rays before and after refraction be i and i' respectively. Then we have

$$p\sin i=h, \quad p'\sin i'=h', \quad h'=h.$$

$\Delta i=i'-i$ is the change of the direction of the ray, and coincides with that in (II-1).

The following quantities

$$L=\cos\Delta i, \quad M=\sin\Delta i/h$$

are of trigonometrical functions, but can be expressed more conveniently as follows. If we put $\overrightarrow{RP}=q$ and $\overrightarrow{RP'}=q'$ in Fig. II-3, we have

$$\begin{aligned} &q^2=p^2-H, \quad q'^2=p'^2-H \\ &pp'L=qq'+H, \quad pp'M=q-q'. \end{aligned} \tag{II-4}$$

In the case of reflection, we take $n'=-n$. The following relation hold betwen L and M

$$L^2+M^2H=1. \tag{II-5}$$

From these formulas, it is seen that L and M are the functions of H only. Using the above results, we derive the relations between the principal or subsidiary variables concerning the ray before and after refraction.

Combining (II-1) with h and h', and using the quantities L and M mentioned above, we obtain

$$g'_{11}=Lg_{11}+Mhg_{21},\quad h'g'_{21}=Lhg_{21}-MHg_{11},\quad h'g'_{31}=hg_{31}$$
$$g'_{12}=Lg_{12}+Mhg_{22},\quad h'g'_{22}=Lhg_{22}-MHg_{12},\quad h'g'_{32}=hg_{32}$$
$$g'_{13}=Lg_{13}+Mhg_{23},\quad h'g'_{23}=Lhg_{23}-MHg_{13},\quad h'g'_{33}=hg_{33}. \qquad \text{(II-6)}$$

From these quantities as well as (I-3) and (I-16), we have the following transformation formulas for the principal and subsidiary variables, If we denote one of the variables E_0 and G_0 by X

$$A'=LA-MB,\qquad E'_X=LE_X+MF_X$$
$$B'=LB+MHA,\quad F'_X=LF_X-MHE_X$$
$$H'=H,\qquad G'_X=G_X. \qquad \text{(II-7)}$$

3. Formulas for Transfer

In this section, several formulas which hold for the ray in passing from one spherical surface to the next are introduced.

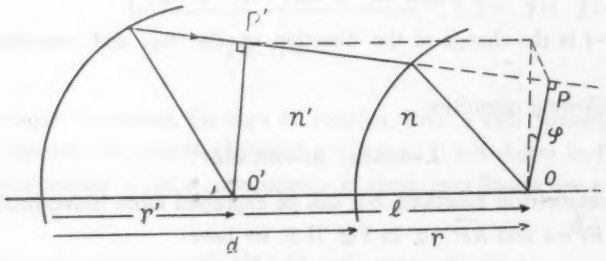

Fig II-4

In Fig. II-4, put $\overrightarrow{O'O}=e$, which quantity will be made use of hereafter. Let the distance between the two spherical surfaces measured on the optical axis be d. Then we have (the dash signifies the quantities for the ν-th surface to discriminate from those of the $\nu+1$th surface).

$$e=n'(d+r-r') \qquad \text{(II-8)}$$

where d has no concern with refractive index.

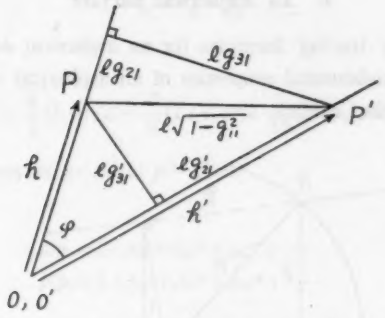

Fig. II-5

Fig. II-5 shows the projections of $\overrightarrow{OP}$ and $\overrightarrow{OP'}$, on a plane perpendicular to $\overrightarrow{OO'}$, on which the projection O and O' are brought one upon the other.

Let the difference in directions of $\overrightarrow{OP}$ and $\overrightarrow{OP'}$, that is the angle between η- and η'-axes, be φ, which was introduced in II-§ 1. η-and ζ-components of e are eg_{21} and eg_{31} respectively. Thus from Fig. II-5 it follows that

$$\sin\varphi=eg'_{31},\ .h=eg_{31}/h',\quad \cos\varphi=(h'-eg'_{21})/h=(h+eg_{21})/h'\ . \qquad \text{(II-9)}$$

This relations enable to derive transfer formulas for the principal and subsidiary variables.

Combining (II-2) with h and h', and using (II-9) we have

$$\begin{aligned}
&g_{11}=g'_{11},\quad hg_{21}=h'g'_{21}-e(1-g'^2_{11}),\quad hg_{31}=h'g'_{31}\\
&g_{12}=g'_{12},\quad hg_{22}=h'g'_{22}+eg'_{11}g'_{12},\quad hg_{32}=h'g'_{32}-eg'_{13}\\
&g_{13}=g'_{13},\quad hg_{23}=h'g'_{23}+eg'_{13}g'_{11},\quad hg_{33}=h'g'_{33}+eg'_{12}\ .
\end{aligned} \qquad \text{(II-10)}$$

Then using (I-3) and (I-16), we obtain the following transfer formulas for the principal and subsidiary variables.

$$\begin{aligned}
&A=A', && E_X=E'_X\\
&D=1-A^2, && F_X=F'_X+eE'_XA'\\
&B=B'+eD', && G_X=G'_X+eE'_X\\
&H=H'+2eB'+e^2D'. &&
\end{aligned} \qquad \text{(II-11)}$$

If calculations for $X=E_0$ and $X=G_0$ are carried out, subsidiary variables can be computed from (I-22) with better accuracy than from the above ray tracing calculation.

4. An Aspherical Surface

To obtain the ray tracing formulas for an aspherical surface, we begin with the investigation of foundamental properties of an aspherical surface on the basis of properties of a spherical surface.

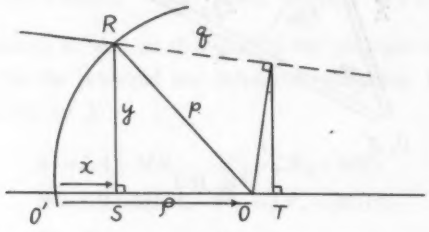

Fig. II-6

In Fig. II-6 which shows an aspherical surface, let the intersection between the optical axis and the normal to the aspherical surface at point R be O, and let the length of the perpendicular $\overline{RS}$ be y which is drawn to the optical axis from R. Put $\overrightarrow{O'O}=\rho$, $\overrightarrow{PO}=p$ and $\overrightarrow{O'S}=x$.

Then

$$p^2-y^2=(\rho-x)^2, \quad (\rho-x)\frac{dx}{dy}=y. \tag{II-13}$$

Denoting quantities for a spherical surface with suffices O in this section, we have in this case $\rho=p=p_0$ and

$$\rho_0^2-y^2=(p_0-x_0)^2. \tag{II-14}$$

The expansion of x_0 in terms of $c_0=(y/p_0)^2$ is given by

$$x_0/p_0=\frac{1}{2}c_0+\frac{1}{8}c_0^2+\frac{1}{16}c_0^3+\frac{5}{128}c_0^4+\cdots\cdots. \tag{II-15}$$

The term of c_0 determines the property of the paraxial ray, but the terms of $c_0^2, c_0^3, \cdots$ have no connection with them.

Regarding these terms as characterizing the aspherical surface, we express the relation between x and c in this case as

$$x/p_0=\frac{1}{2}c+\frac{1}{8}(1+\tau_1)c^2+\frac{1}{16}(1+\tau_2)c^3+\frac{5}{128}(1+\tau_3)c^4+\cdots\cdots \tag{II-16}$$

where $\tau_1, \tau_2, \cdots$ may be regarded as representing the degree of deviation from the sphere. From this formula

$$\frac{1}{\kappa}=\frac{1}{y}\frac{dx}{dy}$$

becomes

$$\frac{1}{\kappa}=1+\frac{1}{2}(1+\tau_1)c+\frac{3}{8}(1+\tau_2)c^2+\frac{5}{16}(1+\tau_3)c^3+\cdots\cdots$$

and (II-13) is expressed as $\rho-x=\kappa$, $p^2-y^2=\kappa^2$.

If we put

$$\begin{aligned}\rho/\rho_0&=1+\rho_1 c+\rho_2 c^2+\rho_3 c^3+\cdots\cdots\\ p/p_0&=1+p_1 c+p_2 c^2+p_3 c^3+\cdots\cdots\end{aligned}\qquad\text{(II-17)}$$

then the above relations give the coefficienents of the formulas (II-17) as follows

$$\begin{aligned}p_1&=-\tfrac{1}{2}\tau_1,\quad 8\rho_2=5\tau_1+2\tau_1^2-3\tau_2\\ 16\rho_3&=-6\tau_1^3+2\tau_1^2+7\tau_2+6\tau_1\tau_2-5\tau_3\\ p_1&=-\tfrac{1}{2}\tau_1,\quad 8p_2=6\tau_1+2\tau_1^2-3\tau_2\\ 16p_3&=-3\tau_1-6\tau_1^2-2\tau_1^3+9\tau_2+6\tau_1\tau_2-5\tau_3\,.\end{aligned}\qquad\text{(II-18)}$$

ρ and p are computed easily and the ray tracing is performed by regarding ρ and p as optical constants if c is known.

Now, c and c_0 are expressed with the subsidiary variables by taking the origin at the centre of the sphere of radius p_0 as

$$\begin{aligned}A^2p_0^2c&=(x-\rho)^2D+2(x-\rho)B+H-C^2\\ A^2p_0^2c_0&=(x-p_0)^2D+2(x-p_0)B+H-C^2\end{aligned}$$

and by subtraction, we obtain

$$C=C_0+\tfrac{1}{4}\tau_1 c^2B'+\cdots\cdots\qquad\text{(II-19)}$$

where B' represents the subsidiary variable B when the origin is taken at O'. The second term is an extremely small quantity in general.

If $\overrightarrow{SO}=\overrightarrow{ST}-\overrightarrow{OT}$ in Fig. II-6 is expressed by subsidiary variables,

$$p_0^2c_0^2=p_0^2-(qA+B)^2\qquad\text{(II-20)}$$

is obtained. Therefore, c can be computed from (II-20) and (II-19).

5. Formulas in the Case when the Surface is a Plane

When a plane surface is considered, the center, on which the origin of coordinates is taken in the case of a spherical surface, becomes indeterminate.

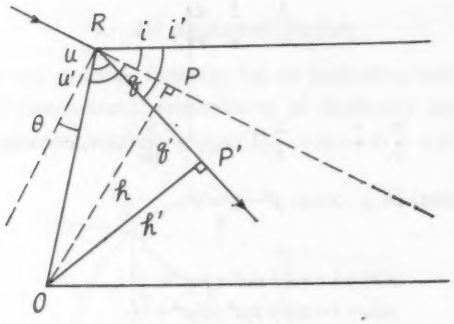

Fig. II-7

In Fig. II-7, let the intersection between the plane and x-axis be the origin O, and that between the ray and the plane be R.

P and P' are dealt with similarly to the case of the spherical surface, in other words i, i', q, q', h and h' are determined in the same way as before. Let the angle between $\overrightarrow{OR}$ and the line of intersection of the incident plane with the plane be θ, and the angles that $\overrightarrow{OR}$ makes with the incident and refracted rays be u and u' respectively. Then we obtain

$$\begin{aligned} &A=\cos i, \quad A'=\cos i', \quad D=\sin^2 i, \quad D'=\sin^2 i' \\ &-qA=B, \quad -q'A'=B' \\ &\cos u=\cos\theta \sin i, \quad \cos u'=\cos\theta' \sin i' . \end{aligned} \tag{II-21}$$

On the other hand, considering $\overrightarrow{OR}$ as a quantity with no regard to refractive index, we have

$$n.\overrightarrow{OR}\cdot\cos u=q, \quad n'.\overrightarrow{OR}\cdot\cos u'=q'$$

which lead to

$$\begin{aligned} &q'/q=n'\cos u'/n\cos u=n'\sin i'/n\sin i=1 \\ &n^2(h'^2+q'^2)=n'^2(h^2+q^2) . \end{aligned} \tag{II-20}$$

From the relations obtained above, formulas for tracing principal and subsidiary variables will be derived.

Put

$$n'/n=\kappa, \quad \varepsilon=A'/A . \tag{II-23}$$

Then from (II-21) and (II-22) we obtain for the relations of D to D' and B to B'

$$\kappa^2 D'=D, \quad B'=\varepsilon B . \tag{II-24}$$

Furthermore, (II-22) gives $H'+B'^2/A'^2=\kappa^2(H+B^2/A^2)$ which becomes by the relation of (I-4) as

$$H'-C'^2=\varepsilon^2\kappa^2(H-C^2)\,. \tag{II-25}$$

We shall consider next the formulas for the principal variables on Fig. II-8.

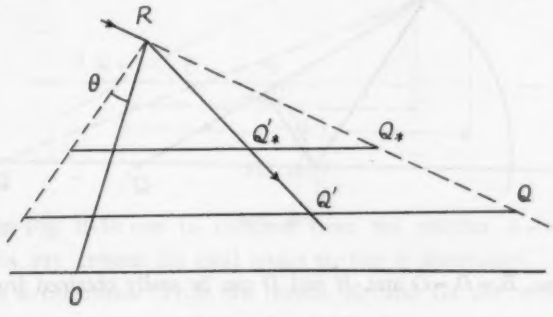

Fig. II-8

In this case QQ' is parallel to x-axis. The expression (I-11) concerning E still holds and moreover $E/E'=n'/n$. Therefore the following relations hold for Q and Q'

$$n^2(s'+q')=n^2(s+q)$$
$$n^2x'/\cos i'=n'^2x^2/\cos i\,.$$

It is seen from the above that the relations for the principal variables are

$$\kappa E'_X=E_X,$$
$$\kappa F'_X=\kappa^2F_X-(1-\kappa^2)BE_X/A$$
$$G'_X=\kappa\varepsilon G_X\,. \tag{II-26}$$

6. Paraxial Ray and Axial Ray

By the use of the ray tracing formulas derived above, formulas for a paraxial ray and an axial ray are obtained.

In the case of a paraxial ray, subsidiary and fundamental variables are zero, which make $q=p$, $q'=p'$ and therefore $L=1$ in (II-4).

From (I-26) and (I-25) it follows that $D=0$, $G_X=F_X$. The ray tracing formulas (II-7) then become

$$M=1/p'-1/p,\quad E'_X=E_X+MG_X,\quad G'_X=G_X\,. \tag{II-27}$$

As for transfer, from (II-11) the ray tracing formulas are given by

$$E_{+X}=E'_X,\quad G_{+X}=G'_X+eE'_X\,. \tag{II-28}$$

Next the method of calculation for an axial ray is considered.

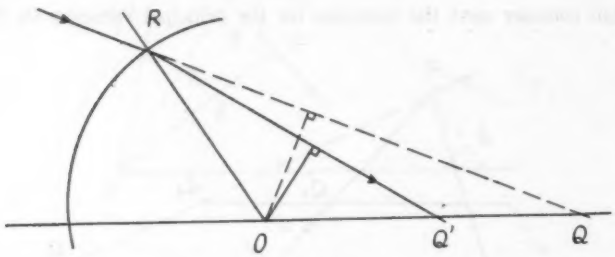

Fig. II-9

In this case, $B_0=D_0=O$ and, H and D can be easily obtained from (I-22) as

$$H=G_G{}^2H_0,\quad D=E_G{}^2H_0\,. \tag{II-29}$$

Then L, M and A are calculated from (II-4). As is shown in Fig. II-9, the object point Q and the image point Q' are on the optical axis, and as explained in Section I-3, $F_G=G_GA$. Thus from (II-7), the ray tracing formulas are

$$E_{+G}=LE_G+MG_GA,\quad G_{+G}=G_G+eE_{+G} \tag{II-30}$$

where the suffix + signifies the quantities for the next surface.

The difference between $-G_G/E_G$ of an axial ray and that of a paraxial ray represents the spherical aberration and the difference between $-1/E_G$ of an axial ray and that of a paraxial ray represents the deviation from sine condition.

In the general case involving skew rays, ray tracing can be performed with only E_X and G_X excepting F_X as treated above, since F_X can be computed from (I-24), (I-26) and (I-27) in case of need.

7. Extra Axial Ray

We treat here the method of obtaining the distortion, sagittal and meridional image surfaces from the ray tracing about an extra axial principal ray.

Using E_E and G_E instead of E_G and G_G dealt with in the previous section, we perform calculations with respect to the pupil. That is, we have instead of (II-29) and (II-30)

$$H=G_E{}^2D_0,\quad D=E_E{}^2D_0$$
$$E_{+E}=LE_E+MAG_E,\quad G_{+E}=G_E+eE_{+E}\,. \tag{II-31}$$

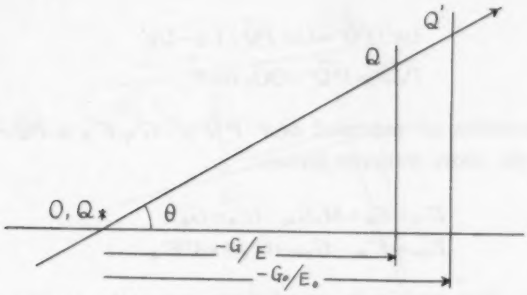

Fig. II-10

Since θ in Fig. II-10 can be obtained from the relation $A=\cos\theta$, the point Q' where the ray crosses the ideal image surface is determined. In other words, the distortion is calculated. From the results, E_G and G_G are obtained by using the formulas

$$AE_{*G}=A_*E_G+MG_G, \quad G_{*G}=G_G+eE_{*G} \tag{II-32}$$

which are derived from (II-7) and (II-11). Since E_G and G_G, which give the position of Q, are computed from the above formulas, and since E_E and G_E for Q_* are already known, Q' and therefore distortion are obtained with better accuracy.

Calculations about E_G and G_G correspond to those of the sagittal image surface as mentioned in Section I-3.

The calculation of the meridional image can be performed with formulas derived below by using the results obtained above.

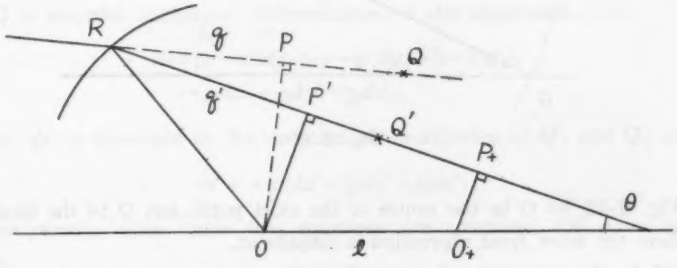

Fig. II-11

In Fig. II-11, let Q and Q' be the meridional object and image points of this surface respectively and, $\overrightarrow{PQ}$ and $\overrightarrow{PQ'}$ be the lengths with no regard to refractive

index, then we have

$$1/n' \cdot \overrightarrow{P'Q'} = 1/n \cdot \overrightarrow{PQ} + 1/q - 1/q'$$
$$\overrightarrow{P_+Q'} = \overrightarrow{P'Q'} - \overrightarrow{OO_+} \cos\theta' .$$

If these quantities are expressed as $n' \cdot \overrightarrow{P'Q'} = -G'_*/E'_*$, $n \cdot \overrightarrow{PQ} = -G_*/E_*$, $1/q' - 1/q = M_*$, then the above formulas become

$$E'_* = E_* + M_* G_*, \quad G'_* = G_*$$
$$E_{*+} = E'_*, \quad G_{*+} = G'_* + eA'E'_* . \tag{II-33}$$

In this case, there are also linear relations among the variables corresponding to the principal variables as seen from the above formulas.

Of cause, for a paraxial ray (II-33) coincides with (II-27) and (II-28).

8. Wave Front Aberration

We shall derive the wave front aberration for the ideal image point of rays from an object point.

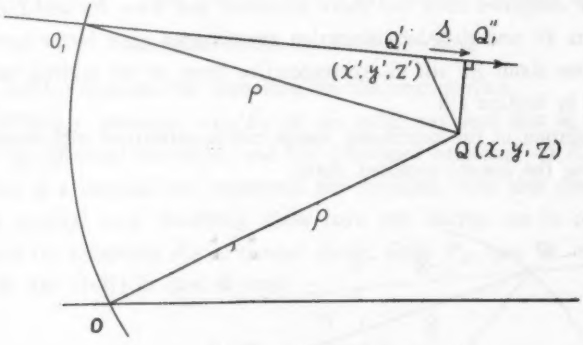

Fig. II-12

In Fig. II-12, let O be the centre of the exist pupil, and Q be the ideal image point where the wave front aberration is considered.

Let O_1 be the intersection between the reference sphere passing through O and having the radius $\overrightarrow{OQ} = \rho$ and the ray passing through the image point Q'_1. Let the perpendicular from Q to the ray be $\overrightarrow{QQ''_1}$. If $\overrightarrow{QQ''_1}$ is a small quantity, we may assume that Q''_1 and Q are at equal distances from O_1.

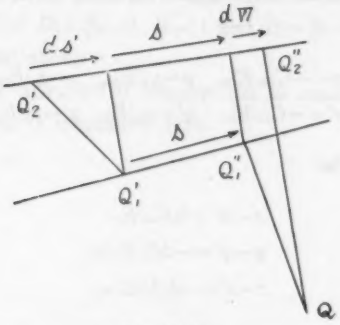

Fig. II-13

Next, take a neighbouring ray $Q'_2Q''_2$ shown in Fig. II-13 and denote

$$\overrightarrow{Q'_1Q''_1}=s,\quad \overrightarrow{Q'_2Q''_2}=s+ds\,.$$

On the ray $Q'_2Q''_2$, let the point on the same wave front as that at Q'_1 be at a distance ds' from Q'_2. Similarly, let the point on the same wave front as that at Q''_1 be at a distance dW from Q''_2. Then we have

$$dW=ds-ds'\,. \tag{II-34}$$

According to this basic idea, we shall find the fundamental formulas expressing the differential dW concerning the wave front on Cartesian co-ordinates. We shall obtain s from $Q(x,y,z)$, $Q'(x',y',z')$ and the direction cosines of the ray g_1, g_2, g_3 in Fig. 13. Since s in the component of $\overrightarrow{Q'_1Q}$ in the direction of $\overrightarrow{Q'_1Q''_2}$ we have

$$s=(x-x')g_1+(y-y')g_2+(z-z')g_3\,. \tag{II-35}$$

If Q is assumed unchanged, differentiation of this expression gives

$$\begin{aligned}ds=&(x-x')dg_1+(y-y')dg_2+(z-z')dg_3\\&-(g_1dx'+g_2dy'+g_3dz')\,.\end{aligned}$$

Next, ds' is expressed as the variation of co-ordinates of Q'_1 and Q'_2 as

$$ds'=-(g_1dx'+g_2dy'+g_3dz')\,.$$

Consequently, when these two equations are combined, dW given by (II-34) becomes

$$dW=(x-x')dg_1+(y-y')dg_2+(z-z')dg_3\,. \tag{II-36}$$

Now, we proceed to express the fundamental formula obtained above in terms of variables so far employed.

Representing E_G and G_G as principal variables for Q'_1, and E_{G0} and G_{G0} for Q, we have

$$x=-G_{G0}/E_{G0},\quad y=y_0/E_{G0},\quad z=z_0/E_{G0}$$
$$x'=-G_G/E_G,\quad y'=y_0/E_G,\quad z'=z_0/E_G\,. \tag{II-37}$$

Let the differences be

$$x-x'=\Delta(G_G/E_G)$$
$$y-y'=-\Delta(1/E_G)y_0$$
$$z-z'=-\Delta(1/E_G)z_0\,. \tag{II-38}$$

Then, expressing g_1, g_2 and g_3 in terms of the principal variables, we modify (I-14) with (II-37), namely

$$g_1=A,\quad t_0g_2=E_Ey_0-E_Gy_{*0},\quad t_0g_3=E_Ez_0-E_Gz_{*0}\,. \tag{II-39}$$

Using these quantities in the above fundamental formula (II-36) and assuming that y_0 and z_0 are constants, we have

$$dW=\Delta(G/E)dA+\Delta(1/E)dp \tag{II-40}$$

where $P=(E_G'b_0-E_Ea_0)/t_0$. When the expressions of (I-21) are used, it is seen that P coincides with P_G in (I-26).

For an axial ray, $p=0$, and (II-40) becomes

$$dW=\Delta(G/E)dA\,. \tag{II-41}$$

Practical value of W can be computed from the following integral by using the results of ray tracing.

$$W=\int\Delta(G/E)dA+\int\Delta(1/E)dP \tag{II-42}$$

or for an axial ray

$$W=\int\Delta(G/E)dA\,. \tag{II-43}$$

Result

Ray tracing formulas as regards principal and subsidiary variables and the formula of wave front aberration are derived with which calculation for skew rays can be dealt with simply.

It is noted that calculation of imaginary rays which have no physical meanings, for example in the case of $D_0=B_0=0$, $H_0<1$ and $D_0=H_0=0$ $B_0\neq 0$, is possible in the same way as mentioned above.

Therefore, complicated character of imagery can be analized to a certain extent by means of the calculation of imaginary rays.

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A Differential Method of Optical Calculation

Part III: Differential Calculation

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Abstract

The differential calculation applicable for obtaining aberration coefficients, which have been derived by other methods, is described.

As mentioned in Part I, subsidiary variables are adequate quantities to define the incident ray, to separate aberration terms and to differentiate the ray tracing formulas.

Therefore, differentiating the ray tracing formulas by the subsidiary variables in the object space, we can derive the differential formulas of ray tracing and use them to calculate the derivatives of principal variables successively from surface to surface.

These derivatives contain terms of higher order of the subsidiary variables for the incident ray, but by using the paraxial quantities in the above derivatives, we can obtain the paraxial differentiation that is the differentiation in the region of Seidel.

However, in this treatise, formulas different from the above ray tracing formulas are used.

By successive differentiations, the 2nd order coefficients and the 3rd order coefficients, the latter of which have scarcely been dealt with in practice, are obtained.

When such differential calculation is extended to the wave front aberration, the differential coefficient thus obtained is the required aberration coefficient.

Further, several methods to separate and compute the aberration terms by performing the calculation on imaginary rays, are briefly referred to.

1. Introduction[1)2)]

General consideration about differential calculation is given in this section.

A quantity, for example the principal variable in the image space represented by X', can be expanded in terms of the subsidiary variables in the object space H_0, B_0 and D_0 as

$$X' = X'_0 + (X'_H H_0 + X'_B B_0 + X'_D D_0) + (\tfrac{1}{2} X'_{H^2} H_0^2 + X'_{HB} H_0 B_0 + \cdots) + \cdots \qquad \text{(III-1)}$$

where X'_0 represents the paraxial value of X'.

1) F. Wachendorf: Optik, **5** (1949) 80.

2) H. A. Buchdahl: *Optical Aberration Coefficients*, (Oxford University Press. London, 1954).

If all the differential coefficients are equal to zero, then $X'=X'_0$ and the image becomes ideal. Hence it is reasonable to regard these coefficients as aberration coefficients of X'.

The differentiation of X' by one of the subsidiary variables, for example H_0, gives

$$dX'/dH_0=X'_H+(X'_{HH}H_0=X'_{HB}B_0+X'_{HD}D_0)+\cdots\cdots. \qquad \text{(III-2)}$$

The derivative obtained from differentiation of ray tracing formulas is of the form given above, and let the differentiation of this form be called the differentiation of ray tracing formulas.

The derivative dX'/dH_0 involves various terms of higher order of subsidiary variables, but if paraxial values are introduced (that means $H_0=B_0=D_0=0$), then

$$dX'/dH_0=X'_H \qquad \text{(III-3)}$$

is obtained. Such a differentiation is called the paraxial differentiation. This treatment corresponds to that in the Seidel's region.

2. Differentiation of Ray Tracing Formulas

Derivatives of principal and subsidiary variables by one of the subsidiary variables in the object space represented by α are derived in the case of a spherical surface.

Since L and M in (II-4) are functions of only H, differentiation of these by H gives

$$\begin{aligned}
&2qq_H=-1,\quad 2q'q'_H=-1\\
&pp'M_H=q_H-q'_H,\quad pp'L_H=q'q_H+qq'_H+1\\
&N_H=M+M_HH
\end{aligned} \qquad \text{(III-4)}$$

where $N=MH$. Then from (II-7) and (II-11) the derivatives of subsidiary variables by α are obtained as

$$\begin{aligned}
&A'_\alpha=LA_\alpha-MB_\alpha+(L_HA-M_HB)H_\alpha\\
&B'_\alpha=LB_\alpha+NA_\alpha+(L_HB+N_HA)H_\alpha\\
&H'_\alpha=H_\alpha
\end{aligned}$$

and

$$\begin{aligned}
&A_{+\alpha}=A'_\alpha,\quad D'_\alpha=-2A'A'_\alpha\\
&B_{+\alpha}=B'_\alpha+eD'_\alpha\\
&H_{+\alpha}=H'_\alpha+2eB'_\alpha+e^2D'_\alpha.
\end{aligned} \qquad \text{(III-5)}$$

By the use of these formulas, it is possible to compute the derivatives $A_{+\alpha}$, $B_{+\alpha}$, $D_{+\alpha}$ and $H_{+\alpha}$ with respect to the next surface from the derivatives A_α, B_α and H_α with respect to this surface.

Next from (II-32) the derivatives of principal variables are

$$AE_{+X\alpha}=A_{+}E_{X\alpha}+MG_{X\alpha}+[E_{+}]_{X\alpha}$$
$$G_{+X\alpha}=G_{X\alpha}+eE_{+X\alpha}$$

where

$$[E_{+}]_{X\alpha}=A_{+\alpha}E_{X}+A_{\alpha}E_{+X}+M_{H}H_{\alpha}G_{X}. \tag{III-6}$$

When the derivatives $E_{X\alpha}$ and $G_{X\alpha}$ for the incident ray to this surface are zero, only $[E_{+}]_{X\alpha}$ remains which may be therefore called the peculiar term to this surface.

3. Paraxial Differentiation

As mentioned in the abstract, when paraxial values are taken in the ordinary differentiation, the differentiation is limited in the region of the 1st order. However, for convenience sake, we employ formulas different from those used in the previous section.

First we take up formulas for H and D from (I-22)

$$H=G^2{}_{E}D_0+2G_{E}G_{G}B_0+G^2{}_{G}(H_0-C^2)+C^2$$
$$D=E^2{}_{E}D_0+2E_{E}E_{G}B_0+E^2{}_{G}(H_0-C^2).$$

Since it is seen from (II-4) and (I-3) that M is a function of only H and A is a function of only D, they may be expanded as

$$M=M_0+M_{H}H+\frac{1}{2}M_{H^2}H^2+\frac{1}{6}M_{H^3}H^3+\cdots\cdots$$
$$pp'M_0=p-p',\quad 2pp'M_{H}=1/p'-1/p$$
$$4pp'M_{H^2}=1/p'^3-1/p^3,\quad pp'M_{H^3}=\frac{3}{8}(1/p'^5-1/p^5)$$
$$A=1-\frac{1}{2}D-\frac{1}{8}D^2-\frac{1}{16}D^3\cdots\cdots. \tag{III-7}$$

From (I-22) the derivatives of subsidiary variables are given by

$$H_{\alpha}=G^2{}_{E}D_{0\alpha}+2G_{E}G_{G}B_{0\alpha}+G^2{}_{G}D_{0\alpha}$$
$$D_{\alpha}=E^2{}_{E}D_{0\alpha}+2E_{E}E_{G}B_{0\alpha}+E^2{}_{G}D_{0\alpha}. \tag{III-8}$$

In practice, there remains only one term among the terms involving $D_{0\alpha}$, $B_{0\alpha}$ and $D_{0\alpha}$, for example, if $\alpha=H_0$, then $B_{0\alpha}=D_{0\alpha}=0$.

Next, by differentiating (III-7), M_{α} and A_{α} are obtained, namely

$$M_{\alpha}=M_{H}H_{\alpha},\quad A_{\alpha}=-\tfrac{1}{2}D_{\alpha}. \tag{III-9}$$

Differentiating (II-32) with the use of these relations and bringing the terms peculiar to this surface together as done in the previous section, we have

$$[E_+]_{X\alpha}=A_{+\alpha}E_X-A_\alpha E_{+X}+M_\alpha G_X\,. \tag{III-10}$$

This is what corresponds to the so-called Seidel's term peculiar to this surface. Then from (II-32), the derivatives of principal variables are derived as follows

$$\begin{aligned}E_{+X\alpha}&=E_{X\alpha}+MG_{X\alpha}+[E_+]_{X\alpha}\\ G_{+X\alpha}&=G_{X\alpha}+\varepsilon E_{+X\alpha}\,.\end{aligned} \tag{III-11}$$

The derivatives of principal variables in image space may be computed by applying these linear type formulas from surface to surface.

According to this reason, if the paraxial derivative of X' by E' is represented by $X'_{E'}$, $X'_{X\alpha}$ can be expressed as the following formula, which is the sum of derivatives relating to all surfaces.

$$X'_{X\alpha}=\sum X'_{E'}[E_+]_{X\alpha}\,. \tag{III-12}$$

It is seen from the above relations that $X'_{X\alpha}$ can be computed only with paraxial values.

4. Second and Third Order Paraxial Differentiations

We proceed to investigate the paraxial differentiation of the 2nd order by applying the results of and the method employed for the 1st order differentiation.

Differentiation of (III-8) by β, which represents one of the subsidiary variables H_0, B_0 and D_0, gives

$$\begin{aligned}\tfrac{1}{2}H_{\alpha\beta}&=G_E\sum(G_{E\alpha}D_{0\beta}+G_{G\alpha}B_{0\beta})\\ &\quad+G_G\sum(G_{E\alpha}B_{0\beta}+G_{G\alpha}H_{0\beta})+\tfrac{1}{2}(1-G^2{}_G)C^2{}_{\alpha\beta}\\ \tfrac{1}{2}D_{\alpha\beta}&=E_E\sum(E_{E\alpha}D_{0\beta}+E_{G\alpha}B_{0\beta})\\ &\quad+E_G\sum(E_{E\alpha}B_{0\beta}+E_{G\alpha}H_{0\beta})-\tfrac{1}{2}E^2{}_GC^2{}_{\alpha\beta}\,.\end{aligned} \tag{III-13}$$

The notation $\sum$ means, for example, $\sum G_{E\alpha}D_{0\beta}=G_{E\alpha}D_{0\beta}+G_{E\beta}D_{0\alpha}$. Then from (III-9)

$$M_{\alpha\beta}=M_HH_{\alpha\beta}+M_{H^2}H_\alpha H_\beta,\quad A_{\alpha\beta}=-\tfrac{1}{2}D_{\alpha\beta}-\tfrac{1}{4}D_\alpha D_\beta \tag{III-14}$$

are derived. From (III-11) the derivatives of principal variables become

$$\begin{aligned}E_{+X\alpha\beta}&=E_{X\alpha\beta}+MG_{X\alpha\beta}+[E_+]_{X\alpha\beta}\\ G_{+X\alpha\beta}&=G_{X\alpha\beta}+\varepsilon E_{+X\alpha\beta}\end{aligned}$$

where

$$\begin{aligned}[E_+]_{X\alpha\beta}&=(A_{+\alpha\beta}E_X-A_{\alpha\beta}E_{+X}-A_{\alpha\beta}E_{+X}+M_{\alpha\beta}G_X)\\ &\quad+\sum(A_{+\alpha}E_{X\beta}-A_\alpha E_{+X\beta}+M_\alpha G_{X\beta})\,.\end{aligned} \tag{III-15}$$

Expressed in sums, it becomes

$$X'_{X\alpha\beta}=\sum X'_{E'}[E_+]_{X\alpha\beta}\,. \tag{III-16}$$

Further, we derive derivatives of the 3rd order employing the results of the 1st and 2nd order differentiations.

Differentiation of (III-13) by γ gives

$$\begin{aligned}\tfrac{1}{2}H_{\alpha\beta\gamma}&=G_E\sum(G_{E\alpha\beta}D_{0\gamma}+G_{G\alpha\beta}B_{0\gamma})\\&+G_G\sum(G_{E\alpha\beta}D+G_{G\alpha\beta}H_{0\gamma})\\&+\sum(G_{E\alpha}G_{E\beta}D_{0\gamma}+G_{E\alpha}G_{G\beta}B_{0\gamma}+G_{G\alpha}G_{G\beta}H_{0\gamma})\\&-\sum G_GG_{G\alpha}C^2{}_{\beta\gamma}\end{aligned}$$

and

$$\begin{aligned}\tfrac{1}{2}D_{\alpha\beta\gamma}&=E_E\sum(E_{E\alpha\beta}D_{0\gamma}+E_{G\alpha\beta}B_{0\gamma})\\&+E_G\sum(E_{E\alpha\beta}B_{0\gamma}+E_{G\alpha\beta}H_{0\gamma})\\&+\sum(E_{E\alpha}E_{E\beta}D_{0\gamma}+E_{E\alpha}E_{G\beta}B_{0\gamma}+E_{G\alpha}E_{G\beta}H_{0\gamma})\\&-\sum E_GE_{G\alpha}C^2{}_{\beta\gamma}\,.\end{aligned} \tag{III-17}$$

Then from (III-14)

$$\begin{aligned}M_{\alpha\beta\gamma}&=M_HH_{\alpha\beta\gamma}+M_{H^2}\sum H_\alpha H_{\beta\gamma}+M_{H^3}H_\alpha H_\beta H_\gamma\\A_{\alpha\beta\gamma}&=-\tfrac{1}{2}D_{\alpha\beta\gamma}-\tfrac{1}{4}\sum D_\alpha D_{\beta\gamma}-\tfrac{1}{8}D_\alpha D_\beta D_\gamma\end{aligned} \tag{III-18}$$

are obtained. By differentiating (III-15), formulas of ray tracing type is derived as

$$\begin{aligned}E_{+X\alpha\beta\gamma}&=E_{X\alpha\beta\gamma}+MG_{X\alpha\beta\gamma}+[E_+]_{X\alpha\beta\gamma}\\G_{+X\alpha\beta\gamma}&=G_{X\alpha\beta\gamma}+eE_{+X\alpha\beta\gamma}\end{aligned} \tag{III-19}$$

where the term peculiar to this surface is

$$\begin{aligned}[E_+]_{X\alpha\beta\gamma}&=(A_{+\alpha\beta\gamma}E_X-A_{\alpha\beta\gamma}E_{+X}+M_{\alpha\beta\gamma}G_X)\\&+\sum(A_{+\alpha\beta}E_{X\gamma}-A_{\alpha\beta}E_{+X\gamma}+M_{\alpha\beta}G_{X\gamma})\\&+\sum(A_{+\alpha}E_{X\beta\gamma}-A_\alpha E_{+X\beta\gamma}+M_\alpha G_{X\beta\gamma})\,.\end{aligned} \tag{III-20}$$

Expression in sum form becomes

$$X'_{X\alpha\beta\gamma}=\sum X'_{E'}[E_+]_{X\alpha\beta\gamma}\,. \tag{III-21}$$

5. Differentiation in the Case when the Surface is a Plane

Formulas for computation in the case of a plane is summerized in this section.

Using (II-24) and (II-23) as the basic formulas, we introduce first the derivatives of subsidiary variables, namely

$$\kappa^2D'=D,\quad A^{12}=1-D',\quad \varepsilon=A'/A\,.$$

Differentiation of these by α gives D'_α, A'_α and ε_α as follows

$$\kappa^2 D'_\alpha = D_\alpha, \quad 2A'A'_\alpha = -D'_\alpha, \quad \varepsilon_\alpha/\varepsilon = A'_\alpha/A'_\alpha - A_\alpha/A . \tag{III-22}$$

Differentiation of the above gives

$$\kappa E_{+X} = E_X, \quad G_{+X} = \kappa\varepsilon G_X + eE_{+X} \tag{III-23}$$

which are obtained by combining (II-26) and (II-11), and denoting the term peculiar to this surface as

$$[G_+]_{X\alpha} = \kappa\varepsilon_\alpha G_X \tag{III-24}$$

we obtain

$$\begin{aligned} &\kappa E_{+X\alpha} = E_{X\alpha} \\ &G_{+X\alpha} = \kappa\varepsilon G_{X\alpha} + eE_{+X\alpha} + [G_+]_{X\alpha} . \end{aligned} \tag{III-25}$$

For paraxial differentiations, if we differentiate (II-23) by α, β and γ employing the derivatives of A given in Section 3 and Section 4, the derivatives of ε are obtained as

$$\begin{aligned} &\varepsilon_\alpha = A_{+\alpha} - A_\alpha, \quad \varepsilon_{\alpha\beta} = A_{+\alpha\beta} - A_{\alpha\beta} - \sum \varepsilon_\alpha A_\beta \\ &\varepsilon_{\alpha\beta\gamma} = A_{+\alpha\beta\gamma} - A_{\alpha\beta\gamma} - \sum \varepsilon_\alpha A_{\beta\gamma} - \sum \varepsilon_{\alpha\beta} A_\gamma . \end{aligned} \tag{III-26}$$

Results of successive differentiations with the use of the above relations are given below.

By differentiation of the 1st order

$$\begin{aligned} &[G_+]_{X\alpha} = \kappa\varepsilon_\alpha G_X, \quad \kappa E_{+X\alpha} = E_{X\alpha} \\ &G_{+X\alpha} = \kappa G_{X\alpha} + eE_{+X\alpha} + [G_+]_{X\alpha} . \end{aligned} \tag{III-27}$$

By differentiation of the 2nd order

$$\begin{aligned} &[G_+]_{X\alpha\beta} = \kappa\varepsilon_{\alpha\beta} G_X + \kappa \sum \varepsilon_\alpha G_{X\beta} \quad \kappa E_{+X\alpha\beta} = E_{X\alpha\beta} \\ &G_{+X\alpha\beta} = \kappa G_{X\alpha\beta} + eE_{+X\alpha\beta} + [G_+]_{X\alpha\beta} . \end{aligned} \tag{III-28}$$

By differentiation of the 3rd order

$$\begin{aligned} &[G_+]_{X\alpha\beta\gamma} = \kappa\varepsilon_{\alpha\beta\gamma} G_X + \kappa \sum \varepsilon_{\alpha\beta} G_{X\gamma} + \kappa \sum \varepsilon_\alpha G_{X\beta\gamma} \\ &\kappa E_{+X\alpha\beta\gamma} = E_{X\alpha\beta\gamma} \\ &G_{+X\alpha\beta\gamma} = \kappa G_{X\alpha\beta\gamma} + eE_{+X\alpha\beta\gamma} + [G_+]_{X\alpha\beta\gamma} . \end{aligned} \tag{III-29}$$

6. Differentiation for Ideal Optical System

The differential expression for the ideal optical system is sought to establish the calculation method to be dealt with as of the case of the standard optical system.

The derivatives of principal variables by α are given from (I-28) as

$$G'_{G\alpha}=E'_{G\alpha}=G'_{E\alpha}=0 \qquad \text{(III-30)}$$

and from (I-30) as

$$2(E'_E-JB_0)E'_{E\alpha}=2JE'_EB_{0\alpha}-J^2(H_{0\alpha}-C^2_\alpha). \qquad \text{(III-31)}$$

The derivatives of subsidiary variables by α are obtained from (I-22) as mentioned before by using the above values.

For paraxial differentiation, (II-30) also holds and the successive derivatives of E'_E are

$$\begin{aligned}
&E'_{E\alpha}=A_\alpha-E'_EA_{0\alpha}\,,\\
&E'_{E\alpha\beta}=A_{\alpha\beta}-E'_EA_{0\alpha\beta}-\sum E'_{E\alpha}A_{0\beta}\,,\\
&E'_{E\alpha\beta\gamma}=A_{\alpha\beta\gamma}-E'_EA_{0\alpha\beta\gamma}-\sum E'_{E\alpha}A_{0\beta\gamma}-\sum E'_{E\alpha\beta}A_{0\gamma}\,.
\end{aligned} \qquad \text{(III-32)}$$

7. Coefficients of Wave Front Aberration

The wave front aberration coefficients of the 1st, 2nd and 3rd orders will be obtained by the use of the fundamental formulas derived in Section II-8 and the results of paraxial differentiation.

Differentials of (II-40) can be expressed with respect to H_0 and B_0 can be expressed as follows:

$$\begin{aligned}
W_H&=\Delta(G_G/E_G)A_H+\Delta(1/E_G)P_H\,,\\
W_B&=\Delta(G_G/E_G)A_B+\Delta(1/E_G)P_B\,.
\end{aligned} \qquad \text{(III-33)}$$

Values of W_H and W_B can be computed from the differentiation of ray tracing, hence W can be obtained from the following integral with respect to H_0 and B_0:

$$W=\int W_H dH_0+\int W_B dB_0\,. \qquad \text{(III-34)}$$

Since Q in Fig. II-12 is the ideal image point, if we use paraxial values in (III-33) we have $W_H=W_B=0$. Hence there is no 1st order differential coefficients of W, and by differentiation of (III-33) with respect to α, we have

$$\begin{aligned}
W_{H\alpha}&=(G_G/E_G)_\alpha A_H\\
W_{B\alpha}&=(G_G/E_G)_\alpha A_B+(1/E_G)_\alpha P_B
\end{aligned} \qquad \text{(III-35)}$$

in which six coefficients relating to the three cases of $\alpha=H_0, =B_0$ and $=D_0$ appear but not the coefficient W_{D0}. W_{HB} is obtained from either one of the above formulas and the relation shown below holds, therefore there are five independent coefficients corresponding to the five terms of Seidel's aberrations.

From (III-9), (III-8) and (I-26) A, P and therefore the wave front aberrations are expressed with the principal variables. In this way the relations

$$
\begin{aligned}
2W_{H^2} &= -E_G G_{GH}, \quad W_{HB} = -E_E G'_{GH} \\
2W_{HB} &= -E_G G_{GB}, \quad W_{B^2} = -E_E G'_{GB} \\
2W_{HD} &= -E_G G_{GD}, \quad W_{BD} = -E_E G'_{GD} \\
W_{D^2} &= 0
\end{aligned} \tag{III-36}
$$

are derived where G' represents the principal variable G when the centre of the pupil is taken as the origin.

Next, differentiation of (III-35) with respect to β gives the following aberration coefficients of the 2nd order:

$$
\begin{aligned}
W_{H\alpha\beta} &= (G_G/E_G)_{\alpha\beta} A_H \\
&\quad + \sum (G_G/E_G)_\alpha A_{H\beta} + \sum (1/E_G)_\alpha P_{H\beta}, \\
W_{B\alpha\beta} &= (G_G/E_G)_{\alpha\beta} A_B + (1/E_G)_{\alpha\beta} P_B \\
&\quad + \sum (G_G/E_G)_\alpha A_{B\beta} + \sum (1/E_G)_\alpha P_{B\beta}.
\end{aligned} \tag{III-37}
$$

Similarly, the 3rd order aberration coefficients are obtained by differentiation of (III-37) with respect to γ thus

$$
\begin{aligned}
W_{H\alpha\beta\gamma} &= (G_G/E_G)_{\alpha\beta\gamma} A_H \\
&\quad + \sum (G_G/E_G)_{\alpha\beta} A_{H\gamma} + \sum (1/E_G)_{\alpha\beta} P_{H\gamma} \\
&\quad + \sum (G_G/E_G)_\alpha A_{H\beta\gamma} + \sum (1/E_G)_\alpha P_{H\beta\gamma}, \\
W_{B\alpha\beta\gamma} &= (G_G/E_G)_{\alpha\beta\gamma} A_B + (1/E_G)_{\alpha\beta\gamma} P_B \\
&\quad + \sum (G_G/E_G)_{\alpha\beta} A_{B\gamma} + \sum (1/E_G)_{\alpha\beta} P_{B\gamma} \\
&\quad + \sum (G_G/E_G)_\alpha A_{B\beta\gamma} + \sum (1/E_G)_\alpha P_{B\beta\gamma}.
\end{aligned} \tag{III-38}
$$

8. Classification and Separation of Aberration Terms

By classifying the wave front aberration terms and aberration terms of principal variables that will be dealt with below and by using the calculation method given heretofore, let us consider the method of separating these terms.

Here (III-1) is again considered, namely

$$
\begin{aligned}
X' = X'_0 &+ (X'_H H_0 + X'_B B_0 + X'_D D_0) + (\tfrac{1}{2} X'_{H^2} H_0^2 + X'_{HB} H_0 B_0 + \cdots) \\
&+ (\tfrac{1}{6} X'_{H^3} H_0^3 + \tfrac{1}{2} X'_{H^2 B} H_0^2 B_0 + \cdots) + \cdots .
\end{aligned}
$$

If all the differential coefficients are equal to zero, then $X' = X'_0$ and the image becomes ideal as mentioned before, and we can regard these coefficients as aberration coefficients which can be classified as arranged in the following diagram.

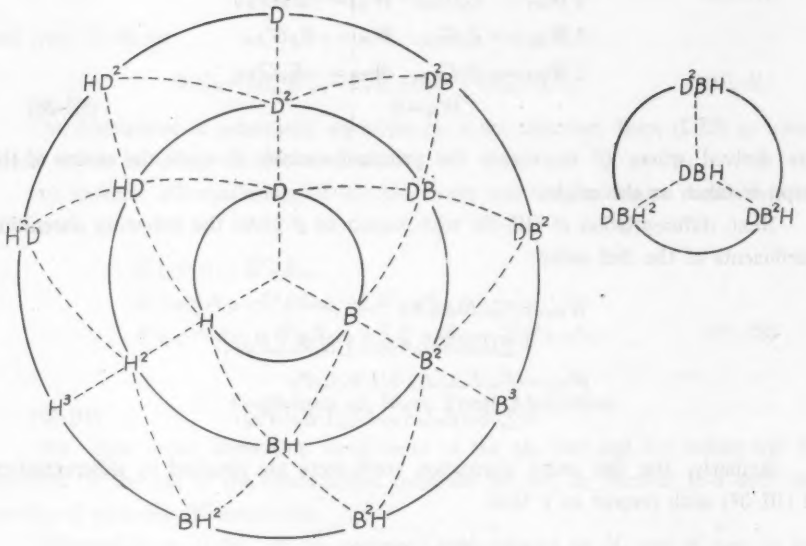

Diagram III-1

In the following, we consider the application of the previously introduced calculation to these aberration terms.

First if X_+ is the value of X when $B_0=D_0=0$ and $H_0>0$, we have

$$X'_+=X'_0+X'_H H_0+\tfrac{1}{2}X'_{H^2}H_0^2+\tfrac{1}{6}X'_{H^3}H_0^3+\cdots\cdots. \qquad \text{(III-39)}$$

Similarly, as to the imaginary ray for which H_0 is negative and has the same absolute value as used above,

$$X'_-=X'_0-X'_H H_0+\tfrac{1}{2}X'_{H^2}H_0^2-\tfrac{1}{6}X'_{H^3}H_0^3+\cdots\cdots \qquad \text{(III-40)}$$

is derived in which X' is represented by X'_-. Addition and subtraction of these two formulas give

$$\begin{aligned}\frac{1}{2}(X'_++X'_-)&=X'_0+\frac{1}{2}X'_{H^2}H_0^2+\frac{1}{24}X'_{H^4}H_0^4+\cdots\cdots\\ \frac{1}{2}(X'_+-X'_-)&=X'_H H_0+\frac{1}{6}X'_{H^3}H_0^3+\frac{1}{120}X'_{H^5}H_0^5+\cdots\end{aligned} \qquad \text{(III-41)}$$

respectively, from which we see that odd and even coefficients can be separated, and further, if we compute X' for several values of H, coefficients of higher order can be separated.

Extending this method of calculation to an imaginary ray such as $D_0=0$,

$B_0 \neq 0$, $H_0 \neq 0$, we can separate the coefficients involving H_0 and B_0.

If the derivative of X' with respect to B_0 represented by dX/dB_0 is computed by the method of ordinary differentiation, the following expression is obtained:

$$dX'/dB_0 = X'_B + X'_{HB}H_0 + \tfrac{1}{2}X'_{H^2B}H_0^2 + \cdots\cdots . \qquad \text{(III-42)}$$

$X'_B, X'_{HB}, X'_{H^2B}, \cdots$ can be separated by the method given above.

By the repetition of such treatment, each group of terms lying on the dotted line in Diagram III-1 can be separated.

It is seen that, in the case of a paraxial differentiation, coefficients X'_H, X'_B and X'_D on the 1st zone are determined by the differential calculation of the 1st order, $X'_{H^2}, X'_{HB}, \cdots$ on the 2nd zone by the differential calculation of the 2nd order and so on.

As stated before, there exists neither the coefficient of the wave front aberration concerning only D_0 nor the coefficient of the 1st order corresponding to the term on the 1st zone in Diagram III-1. Accordingly there appear five aberration coefficients of the 1st order, nine coefficients of the 2nd order and fourteen coefficients of the 3rd order corresponding to the terms on the 2nd, 3rd and fourth zones respectively.

Results

Two kinds of differential calculation methods are derived.

The first that is the differentiation of ray tracing formulas is available to obtain detailed information about the complicated aberrations of an optical system.

The second that is the paraxial differentiation is used to obtain aberration coefficients of the quantity suited to describe the imagery directly as the differential coefficients.

It seems difficult to use such a differential method with respect to the ordinary ray tracing formulas which are generally being used.

A Differential Method of Optical Calculation
Part IV: Differential

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Abstract

For calculation of correction on an optical system, various methods of calculation, which depend on the nature of the correction, are being used.

In this treatise, a synthesized differential method of calculation is considered which is applicable to the cases of an aspherical surface, an eccentric optical system, displacements of object plane and entrance pupil, and so on.

The case of differentiation of ray tracing formula and the case of paraxial differentiation are dealt with both involving skew rays.

When any part of the optical system deviates by a small amount, the principal and subsidiary variables in this space change and the principal variables in image space also change by the amount determined by the partial differential relations among these variables.

Although, for this partial differential calculation, the method, by which the differential coefficients of principal variables in image space with respect to variables in any space successively from image to object space is computed, is commonly being used, here we compute the coefficients directly using the results of differential calculations obtained in Part III.

After this treatment, calculation formula of differential correction is derived in the case in which surface elements vary, or in other words, optical constants vary.

Next for an eccentric optical system, in which the center of any surface deviates from the optical axis, and for an aspherical surface, the surface which was originally spherical, formulas of corrections are derived.

Further, the correction formulas for an optical system, in which the positions of object plane and entrance pupil are displaced, are obtained.

Lastly, formulas of differential correction for the wave front aberration concerning the above various variations are expressed in general form.

1. Introduction

Partial differential calculations have been employed for corrections of an optical system in general.[1)2)]

First, the principle of the method of obtaining the relation between partial differential calculation and correcting calculation, is considered by employing the notations hitherto used.

1) F. D. Cruickshank: Proc. Phys. Soc. **57**, (1945) 350.
2) Tatsuro Suzuki: Text Book of Kogaku Konwa Kai (1953).

Derivatives of X' by X is expressed as follows in terms of the partial derivatives of X' with respect to E' and G' which are the principal variables of the emerging ray of the νth surface.

$$X'_X = X'_{E'} E'_X + X'_{G'} G'_X . \tag{IV-1}$$

When a quantity V of a part of the optical system is varied by a small amount dV, an emerging ray and finally X'_X are changed by the amount defined by the following relation

$$X'_{XV} = X'_{E'} E'_{XV} + X'_{G'} G'_{XV} + X'_{E'V} E'_X + X'_{G'V} G'_X$$

where

$$X'_{E'V} = X'_{E'H'} H'_V + X'_{E'B'} B'_V + X'_{E'D'} D'_V$$
$$X'_{G'V} = X'_{G'H'} H'_V + X'_{G'B'} B'_V + X'_{G'D'} D'_V \tag{IV-2}$$

which are derived from the differentiation of (IV-1) by V.

Therefore, in ordinary differentiation, X'_{XV} can be computed when the derivatives of X' and those with respect to V are obtained in the above formulas.

For a paraxial ray, H'_V, B'_V and D'_V in (IV-2) become zero, then

$$X'_{XV} = X'_{E'} E'_{XV} + X'_{G'} G'_{XV} \tag{IV-3}$$

is derived.

As for paraxial differentiation to obtaine $X'_{X\alpha}$, the derivative of X'_X by V, we differentiate first (IV-1) by α, thus

$$X'_{X\alpha} = X'_{E'} E'_{X\alpha} + X'_{G'} G'_{X\alpha} + X'_{E'\alpha} E'_X + X'_{G'\alpha} G'_X$$

where

$$X'_{E'\alpha} = X'_{E'H'} H'_\alpha + X'_{E'B'} B'_\alpha + X'_{E'D'} D'_\alpha$$
$$X'_{G'\alpha} = X'_{G'H'} H'_\alpha + X'_{G'B'} B'_\alpha + X'_{G'D'} D'_\alpha . \tag{IV-4}$$

Then the differentiation of the above by V gives $X'_{X\alpha V}$, namely

$$X'_{X\alpha V} = X'_{E'} E'_{X\alpha V} + X'_{G'} G'_{X\alpha V} + X'_{E'\alpha} E'_{XV} + X_{G'\alpha} G'_{XV} + X'_{E'\alpha V} E'_X + X'_{G'\alpha V} G'_X$$

where

$$X'_{E'\alpha V} = X'_{E'H'} H'_{\alpha V} + X'_{E'B'} B'_{\alpha V} + X'_{E'D'} D'_{\alpha V}$$
$$X'_{G'\alpha V} = X'_{G'H'} H'_{\alpha V} + X'_{G'B} B'_{\alpha V} + X'_{G'D'} D'_{\alpha V} . \tag{IV-5}$$

These formulas coincide with what are derived from the differentiation of (IV-2) by α.

The calculation method of the above partial differential coefficients of X' will be dealt with in the following section.

2. Partial Differential Calculation

The method, by which the partial differential calculations of principal variables

are carried out by the use of the results of differential calculations of ray tracing, will be considered first.

Using the relation of (I-19), namely

$$G_{G0}E_{E0}-G_{E0}E_{G0}=A/A_0 \tag{IV-6}$$

and solving the equation of (IV-1) for $X=E_0$ and $X=G_0$ as

$$\begin{aligned} X'_{E0}&=X'_G G_{E0}+X'_E E_{E0}\\ X'_{G0}&=X'_G G_{G0}+X'_E E_{G0} \end{aligned} \tag{IV-7}$$

we have

$$\begin{aligned} X'_E&=\frac{A_0}{A}(G_{G0}X'_{E0}-G_{E0}X'_{G0})\\ X'_G&=\frac{A_0}{A}(E_{E0}X'_{G0}-E_{G0}X'_{E0}). \end{aligned} \tag{IV-8}$$

The right terms of these formulas can be computed from the previous differential calculation. For a paraxial ray, $A=A_0=1$ and the above formulas become

$$\begin{aligned} X'_E&=G_{G0}X'_{E0}-G_{E0}X'_{G0}\\ X'_G&=E_{E0}X'_{G0}-E_{G0}X'_{E0}. \end{aligned} \tag{IV-9}$$

The partial derivatives by subsidiary variables are obtained similarly. To do this, (IV-1) is differentiated and the derivative of X' by X and α is expressed through E, G, H, B and D as

$$X'_{X\alpha}=X'_E E_{X\alpha}+X'_G G_{X\alpha}+X'_{E\alpha}E_X+X'_{G\alpha}G_X$$

where

$$\begin{aligned} X'_{E\alpha}&=X'_{EH}H_\alpha+X'_{EB}B_\alpha+X'_{ED}D_\alpha\\ X'_{G\alpha}&=X'_{GH}H_\alpha+X'_{GB}B_\alpha+X'_{GD}D_\alpha. \end{aligned} \tag{IV-10}$$

$X'_{E\alpha}$ and $X'_{G\alpha}$, which will be used later on, are obtained as follows.

If we put

$$[X']_{X\alpha}=X'_{X\alpha}-X'_E E_{X\alpha}-X'_G G_{X\alpha} \tag{IV-11}$$

this is a quantity obtainable by the calculations mentioned before, with which (IV-10) for the cases of $X=E_0$ and $X=G_0$ are expressed as follows

$$\begin{aligned} X'_{E\alpha}E_{E0}+X'_{G\alpha}G_{E0}&=[X']_{E0\alpha}\\ X'_{G\alpha}G_{G0}+X'_{E\alpha}E_{G0}&=[X']_{G0\alpha}. \end{aligned}$$

By combining these two formulas and using the relation of (I-19), we have

$$\begin{aligned} X'_{E\alpha}&=\frac{A_0}{A}(G_{G0}[X']_{E0\alpha}-G_{E0}[X']_{G0\alpha})\\ X'_{G\alpha}&=\frac{A_0}{A}(E_{E0}[X']_{G0\alpha}-E_{G0}[X']_{E0\alpha}). \end{aligned} \tag{IV-12}$$

Next, for $\alpha=H_0, B_0$ and D_0, the formula of $X'_{E\alpha}$, which is the first formula of (IV-10), becomes

$$\begin{aligned} &X'_{EH}H_{H0}+X'_{EB}B_{H0}+X'_{ED}D_{H0}=X'_{EH0}\\ &X'_{EB}H_{B0}+X'_{EB}B_{B0}+X'_{ED}D_{B0}=X'_{EB0}\\ &X'_{EH}H_{D0}+X'_{EB}B_{D0}+X'_{ED}D_{D0}=X'_{ED0}. \end{aligned} \tag{IV-13}$$

X'_{EH}, X'_{EB} and X'_{ED} (similarly X'_{GH}, X'_{GB} and X'_{GD}) are obtained by solving these simultaneous equations. For paraxial differentiation, the following relation derived from (I-22) is used to solve the above equations:

$$\begin{vmatrix} H_{H0} & B_{H0} & D_{H0}\\ H_{B0} & B_{B0} & D_{B0}\\ H_{D0} & H_{D0} & H_{D0} \end{vmatrix}=1. \tag{IV-14}$$

The derivatives by E', G', H', B' and D' concerning the emerging ray are obtained in the same was as above by using $E'=E_+$, $G'=G$, $D'=D_+$ and $H'=H$ and calculating B'_α from (I-22) similarly to the case of H_α and D_α.

3. Preliminary Calculation

If the distance from the origin of one surface to the origin of the next surface is varied, the relation of the incident ray to the latter surface will be changed.

The derivative of $X'_X=X'_{G+}G_{+X}+X'_{E+}E_{+X}$ of (IV-1) by e is

$$\begin{aligned} X'_{Xe}=&X'_{G+}G_{+Xe}+X'_{G+}G_{+Xe}+X'_{XH+}H_{+e}+X'_{XB+}B_{+e}\\ &+X'_{XD+}D_{+e} \end{aligned}$$

where

$$\begin{aligned} &X'_{XH+}=X'_{G+H+}G_{+X}+X'_{E+H+}E_{+X}\\ &X'_{XB+}=X'_{G+B+}G_{+X}+X'_{E+B+}E_{+X}\\ &X'_{XD+}=X'_{G+D+}+G_{+X}+X'_{E+D+}E_{+X}. \end{aligned} \tag{IV-15}$$

Since the derivatives of X' in the right-hand side of (IV-15) are obtained by the previous method, X'_{Xe} can be computed if the derivatives by e are known.

Differentiation of (II-11) by e gives

$$\begin{aligned} &G_{+Xe}=E_{+X}, \quad E_{+Xe}=0\\ &H_{+e}=2B_+, \quad B_{+e}=D_+, \quad D_{+e}=0. \end{aligned} \tag{IV-16}$$

Using these quantities, we obtain the required relation

$$X'_{Xe}=X'_{G+}E_{+X}+2X'_{XH+}B_++X'_{XB+}D_+. \tag{IV-17}$$

For a paraxial ray, $B_+=D_+=0$ and the above becomes

$$X'_{Xe}=X'_{G+}E_{+X}\,. \tag{IV-18}$$

In the case of paraxial differentiation, we obtain $X'_{X\alpha}$ and differentiate it by e, but the result is the same when (IV-17) is differentiated by α as explained in Introduction, therefore we have

$$X'_{X\alpha e}=X'_{G+}E_{+X\alpha}+X'_{G+\alpha}E_{+E}+2X'_{XH+}B_{+\alpha}+X'_{XB+}D_{+\alpha}\,, \tag{IV-19}$$

Next, the variations of the optical constants concerning the surface are expressed as the differentials of L and M, but there exists the relation $L^2+M^2H=1$ of (II-5), the differentiation of which gives

$$LL_M=-MH\,. \tag{IV-20}$$

If the formulas of (II-7) are regarded as expressing the relations between the variables about the rays before and after passing the surface and if they are differentiated by L, M and the above quantity L_M is used. we have

$$\begin{aligned} &G'_{XM}=0 \quad AE'_{XM}=E_XA'_M+G'_X\\ &H'_M=0,\ B'_M=H'A'/L.\ A'_M=-B'/L,\ D'_H=-2A'A'_M\,. \end{aligned} \tag{IV-21}$$

The derivative of X'_X by M is expressed through the variables about the emerging ray from this surface as

$$\begin{aligned} X'_{XM}=&X'_{G'}G'_{XM}+X'_{E'}E'_{XM}+X'_{XH'}H'_M\\ &+X'_{XB'}B'_M+X'_{XD'}D'_M \end{aligned}$$

where

$$\begin{aligned} X'_{XH'}&=X'_{E'H'}E'_X+X'_{G'H'}G'_X\\ X'_{XB'}&=X'_{E'B'}E'_X+X'_{G'B'}G'_X\\ X'_{XD'}&=X'_{E'D'}E'_X+X'_{G'D'}G'_X \end{aligned} \tag{IV-22}$$

and from (IV-21) and (IV-22), the required differential formula is obtained as

$$X'_{XM}=X'_{E'}(E_XA'_M+G'_X)/A+(X'_{XB'}H'+2X'_{XD'}B')A'/L\,. \tag{IV-23}$$

For a paraxial ray, since the subsidiary variables do not take part in the differentiation,

$$X'_{XM}=X'_{E'}G'_X\,. \tag{IV-24}$$

The paraxial differentiation is the differentiation of (IV-23) by α, namely

$$\begin{aligned} X'_{X\alpha M}=&X'_{E'}(G'_{X\alpha}-B'_\alpha E_X-G'_XA_\alpha)+X'_{E'\alpha}G'_X+X'_{XB'}H'_\alpha\\ &+2X'_{XD'}D'_\alpha\,. \end{aligned}$$

In this case the variation of the surface involves the variation of M_H in (III-9), and therefore the differentiation by M_H gives the following derivative

$$X'_{\alpha M H}=X'_{E'}E'_{\alpha M H}=X'_{E'}G'_{X}H'_{\alpha}. \qquad \text{(IV-26)}$$

Relations derived in this section will be widely applied hereafter to the calculation of corrections.

4. Differential Corrections for Surface Elements

The radius of a surface, the refractive index of a space and differential variations of e, M and M_H, which are the quantities derived from the surface elements, will be considered for obtaining differential variation of X'_X with respect to the above quantities by using the results of the previous section. X'_X by the above surface elements will be obtained from the results in the previous section.

First, differentiations of (II-8) by α, n', γ and γ_+ give

$$e_\alpha=n', \quad e_{n'}=e/n', \quad e_\gamma=-n', \quad e_{\gamma+}=n'. \qquad \text{(IV-27)}$$

Differentials of (II-4) by γ, n and n', namely the derivatives of M by these quantities, are given by

$$\begin{aligned} &pp'M_\gamma=np/q-n'p'/q'-2nn'\gamma M \\ &pp'M_n=\gamma(p/q-Mp'), \quad pp'M_{n'}=-\gamma(p/q-Mp). \end{aligned} \qquad \text{(IV-28)}$$

For a paraxial ray, we put $p=q$, $p'=q'$ in the above formulas, which become

$$M_\gamma=-M/\gamma, \quad M_n=1/np, \quad M_{n'}=-1/n'p'. \qquad \text{(IV-29)}$$

Differentiation of (III-7) gives the derivatives of M_H expressed as

$$\begin{aligned} &2pp'M_{H\gamma}=-\frac{3}{\gamma}M, \quad 2npp'M_{Hn}=2/p-1/p' \\ &2n'pp'M_{Hn'}=1/p-2/p'. \end{aligned} \qquad \text{(IV-30)}$$

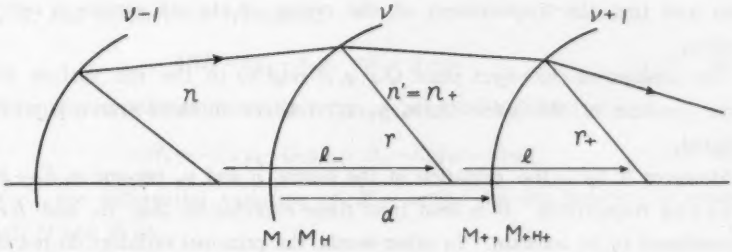

Fig. IV-1

In Fig. IV-1, the relations between the quantities in front and behined the νth surface, for which variations are considered, are shown. The variation of d has effect upon e, that of γ upon e_-, e_+, M and M_H, that of n' upon e, M, M_+, M_H and M_{+H_+}. Therefore in an ordinary case, using the quantities of (IV-17), (IV-23), (IV-27) and (IV-28), we obtain

$$X'_{Xd}=X'_{Xe}e_a,\quad X'_{X\gamma}=(X'_{Xe}e_{\gamma+})_--X'_{Xe}e_\gamma+X'_{XM}M_\gamma$$
$$X'_{Xn'}=X'_{Xe}e_{n'}+X'_{XM}M_{n'}+(X'_{XM}M_n)_+ \qquad \text{(IV-31)}$$

where $(\quad)_-$ and $(\quad)_+$ are the quantities concerning the ν-1th and the ν+1th surface respectively.

For paraxial differentiation, similarly to the above, we obtain the differential correction formulas for X'_X as

$$X'_{Xad}=X'_{Xae}ed\,,$$
$$X'_{Xa\gamma}=(X'_{Xae}e_{\gamma+})_-+X'_{Xae}e_\gamma+X'_{XaM}M_\gamma+X'_{XaMH}M_{H\gamma}\,,$$
$$X'_{Xan'}=X'_{Xae}e_{n'}+X'_{XaM}M_{n'}+(X'_{XaM}M_n)_+$$
$$+X'_{XaMH}M_{Hn'}+(X'_{XaMH}M_{Hn'})_+\,. \qquad \text{(IV-32)}$$

5. Eccentric Optical System

From the results of calculations concerning the partial derivatives in Section IV-2, formulas for calculation of derivatives of X'_X are derived for the case in which the center of one of the spherical surfaces in the optical system is displaced from the optical axis.

Consider a co-ordinate system with the centre at the centre O of νth surface of the optical system described in Section I-1. For a general treatment of an eccentric optical system, it is sufficient if we consider that the fundamental plane varies and that the displacement of the centre of the νth surface is only in y-direction.

The position of the object point $Q(x, y, z)$ relative to the νth surface as well as the position of the point $Q_*(x_*, y_*, z_*)$ relative to the entrance pupil remain unaltered.

However if Δy is the deviation of the centre, y and y_* become $y_0/E_G+\Delta y$ and $y_{*0}/E_E+\Delta y$ respectively. It is seen from these expressions that E_G and E_E may be considered to be constant. In other words, the principal variables do not change with the deviation of the centre from the optical axis.

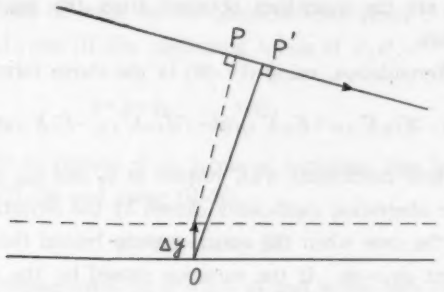

Fig. IV-2

Among the subsidiary variables, D depends only on the direction of the ray and is unchanged, but H and B vary because the perpendicular $\overrightarrow{OP}$ from O in Fig. IV-2 changes.

The variations of H and B are calculated by the use of (I-14) as

$$\begin{aligned} t^2H&=(x^2+a)(x_*^2+c)-(xx_*+b)^2\,,\\ t^2B&=-x_*a+(xx_*+b)-xc\,,\\ a&=y^2+z^2,\quad b=yy_*+zz_*,\quad c=y_*^2+z_*^2\,, \end{aligned} \tag{IV-33}$$

and the altered values of variables in image space are computed with ray tracing formula.

For differential correction, expressing the deviation as a differential $dy=dy_*$ and using x, x_*, t which are unchanged, we obtain by differentiating the above expressions with respect to y or y_*

$$\begin{aligned} \tfrac{1}{2}t^2H_y&=(x_*^2+c)y-(xx_*+b)(y+y_*)+(x^2+a)y_*\\ t^2B_y&=(x-x_*)(y-y_*)\,. \end{aligned} \tag{IV-34}$$

In the case of ordinary differentiation these quantities are computed from the relations in Section I-5. Then for paraxial differentiation, we put $a=b=c=0$ and $x-x_*=t$, and we have for H_y and B_y

$$\tfrac{1}{2}tH_y=xy_*-x_*y,\quad tB_y=y-y_*\,. \tag{IV-35}$$

They may also be expressed in terms of principal variables as

$$\tfrac{1}{2}H_y=-G_Fy_0+G_Gy_{*0},\quad B_y=E_Gy_{*0}-E_Fy_0\,. \tag{IV-36}$$

X'_{xy}, the differential variation of X'_x caused by the deriation, is expressed through H and B as

$$X'_{xy}=X'_{xH}H_y+X'_{xB}B_y\,. \tag{IV-37}$$

X'_{xH} and X'_{xB} are the quantities obtained from the partial differential calculations stated before.

For paraxial differentiation, using (IV-36) in the above formula, we obtain

$$X'_{xy}=(-2G_E X'_{xH}+E_E X'_{xB})y_0+(2G_G X'_{xH}-E_G X'_{xB})y_{*0}. \quad \text{(IV-38)}$$

Calculation of these coefficients with respect to y_0 and y_{*0} gives the quantities corresponding to the aberration coefficients caused by the deviation of the centre.

The above is the case when the optical system behind the νth surface is displaced by the amount $dy_\nu=dy$. If the variation caused by the displacement $dy_{\nu+1}=-dy$ of the optical system behind the $\nu+1$th surface is added to the above, the variation of X'_x caused by the displacement of only one spherical surface is given. Thus

$$X'_{xy}=X'_{xy\nu}-X'_{xy\nu+1}. \quad \text{(IV-39)}$$

Besides, the variation of y' and y'_* in image space are expressed by the combination of quantities corresponding to the magnification and focal length of the paraxial ray

$$y'_y=(E'_{\nu G}-E'_{\nu+1 G})/E'_G$$
$$y'_{*y*}=(E'_{\nu E}-E'_{\nu+1 E})/E'_E. \quad \text{(IV-40)}$$

6. Aspherical Surface Modification

With the use of the results of partial differentiation and fundamental relations for an aspherical surface derived in Section II-4, differential formulas in the case in which any sphere constituting the optical system is subjected to an infinitesimal change and becomes aspherical, but retains the rotational symmetry about the optical axis, will be considered.

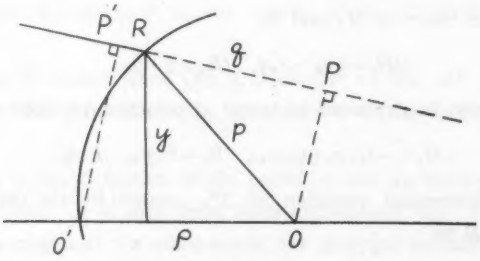

Fig. IV-3

The discussion will be made on the practical stand-point.

First from (II-17) and (II-18), neglecting terms of $c^2, c^3, \cdots$, we have

$$\rho = p = p_0 - \frac{1}{2p_0}\tau_1 y^2. \tag{IV-41}$$

Next we proceed to express y^2 in terms of variables that have been employed. By taking $c=c_0$ of (II-19) and using (II-20),

$$y^2 = p_0^2 - (qA+B)^2 \tag{IV-42}$$

is obtained. When transformed by $q^2 = p_0^2 - H$ and when only the terms of the 1st order is taken, the above becomes

$$y^2 = H - 2p_0 B + p_0^2 D\,.$$

By comparing this with (II-11), we see that y^2 corresponds to H in the case in which the origin is displaced by a distance $-p_0$ and brought to O'. That is to say, taking $\overrightarrow{O'P'^2}$ in the above figure as H'_0, we have

$$y^2 = H'^0\,. \tag{IV-43}$$

As is shown in (IV-41), ρ and p are subjected to the same change in this aspherical surface. This corresponds to the variation of the radius in the case of sphere. Therefore, $-\dfrac{1}{2np_0} H'_0 d\tau$ is the quantity that corresponds to the variation dr of the radius in the case of sphere.

Using this relation, we obtain the variation of X' against τ as .

$$X'_{X\tau} = -\frac{1}{2np_0} X'_{X_I} H'_0. \tag{IV-44}$$

For paraxial differentiation, it becomes

$$X'_{X\sigma\tau} = -\frac{1}{2np_0} X'_{X_I} H'_{0\sigma}\,. \tag{IV-45}$$

This is the differential expression corresponding to the variation of Seidel's aberration term.

7. Displacements of Object Plane and Entrance Pupil

The differential relations in thc case, in which the axial point of the object plane and the entrance pupil of the ideal optical system treated in Section I-9 are displaced, will be considered.

The relations among variables in the ideal optical system depend only on J as can be seen from (I-28) and (I-29).

First, by differentiating these, we obtain

$$G'_{GJ}=0,\quad E'_{GJ}=1,\quad G'_{EJ}=0$$
$$(E'_E-JB_0)E'_{EJ}=E'_EB_0-F(H_0-C^2)\,. \tag{IV-46}$$

For subsidiary variables, differentiation of (I-28) with the use of (III-9) and (I-28) gives the following expressions

$$H'_J=0,\quad B'_J=E'_{EJ}B_0-H_0+C^2$$
$$\tfrac{1}{2}D'_J=-E'_EE'_{EJ}A_0^2\,. \tag{IV-47}$$

In the case of paraxial differentiation, the differentiations of (I-28) and (I-29) by α lead to

$$G'_{GJ}=G'_{EJ}=E'_{EJ}=0,\quad E'_{GJ}=1\,,$$
$$G'_{G\alpha J}=E'_{G\alpha J}=G'_{E\alpha J}=0,\quad E'_{E\alpha J}=B_{0\alpha}-JH_{0\alpha}\,,$$
$$H'_{\alpha J}=0\,,\quad B'_{\alpha J}=-H_{0\alpha}\,,\quad \tfrac{1}{2}D'_{\alpha J}=-E'_{E\alpha J}\,. \tag{IV-48}$$

For $\alpha=D_0$, the derivatives by α become zero.

The displacements of the object plane and the entrance pupil are now considered on Fig. IV-4 which corresponds to Fig. I-6 of the ease of the ideal optical system.

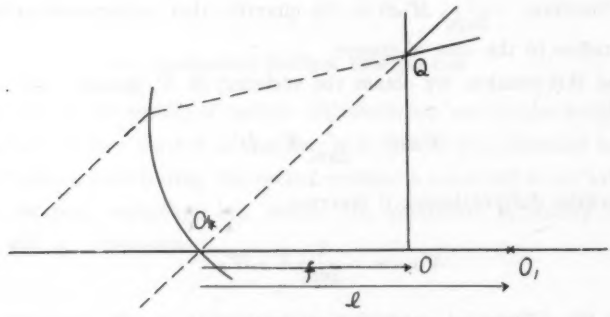

Fig. IV-4

Since the displacement of the object plane corresponds to the variation of $J(=1/f)$, differentiation of $X'_X=X'_{E'}E'_X+X'_{G'}G'_X$ by J gives

$$X'_{XJ}=X'_{G'}G'_{XJ}+X'_{E'}E'_{XJ}+X'_{XH'}H'_J$$
$$+X'_{XB'}B'_J+X'_{XD'}D'_J\,. \tag{IV-49}$$

Using (IV-46) and (IV-47) in this formula, we obtain X'_{XJ}.

For paraxial differentiation, differentiation of (IV-49) by α gives

$$X'_{xaJ}=X'_{G'}G'_{xaJ}+X'_{E'}E'_{xaJ}$$
$$+X'_{xH'}H'_{aJ}+X'_{xB'}B'_{aJ}+X'_{xD'}D'_{aJ}$$
$$+X'_{xH'J}H'_{a}+X'_{xB'J}B'_{a}+X'_{xD'J}D'_{a}$$

where

$$X'_{xH'J}=X'_{E'H'}E'_{xaJ}+X'_{G'H'}G'_{xJ}$$
$$X'_{xB'J}=X'_{E'B'}E'_{xJ}+X'_{G'B'}G'_{xJ}$$
$$X'_{xD'J}=X'_{E'D'}E'_{xJ}+X'_{G'D'}G'_{xJ}\,. \qquad \text{(IV-50)}$$

8. Differential Correction of Wave Front Aberration

If the quantity concerning differential correction, for example, the surface element, eccentric deviation, coefficient of an aspherical surface, is expressed by V, the derivative of X'_x by V that is X'_{xV} and the paraxial derivative X'_{xaV} are obtained by the methods mentioned previously.

For ordinary differentiation, the following formula of (II-42)

$$W=\int\Delta(G_G/E_G)dA+\int\Delta(1/E_G)dP$$

is used. For differentiation of $\Delta(G_G/E_G)$ and $\Delta(1/E_G)$, differentiation of a paraxial ray is used at the same time.

With the differential notations hitherto used, the differentiation of W is expressed as

$$W_V=\int\Delta(G_G/E_G)_V dA+\int\Delta(1/E_G)_V dP$$
$$+\int\Delta(G_G/E_G)dA_V+\int\Delta(1/E_G)dP_V \qquad \text{(IV-51)}$$

from which W_V can be computed.

For paraxial differentiation, we have from the formula of (III-35)

$$W_{Ha}=(G_G/E_G)_a A_H\,,$$
$$W_{Ba}=(G_G/E_G)_a A_B+(1/E_G)_a P_B \qquad \text{(IV-52)}$$

and their derivatives by V are given by

$$W_{HaV}=(G_G/E_G)_{aV}A_H+(G_G/E_G)_a A_{HV}\,,$$
$$W_{BaV}=(G_G/E_G)_{aV}A_B+(G_G/E_G)_a A_{BV}$$
$$+(1/E_G)_{aV}P_B+(1/E_G)_a P_{BV}\,. \qquad \text{(IV-53)}$$

For eccentric deviation in the case of paraxial differentiation, X'_{xy} has been obtained already, hence we begin with the following formula of (III-33)

$$W_H=\Delta(G_G/E_G)A_H\,,$$
$$W_B=\Delta(G_G/E_G)A_B+\Delta(1/E_G)P_B\,. \qquad \text{(IV-54)}$$

The differentiations of these by y gives

$$W_{Hy}=(G_G/E_G)_y A_H\,,$$
$$W_{By}=(G_G/E_G)_y A_B+(1/E_G)_y P_B\,. \qquad \text{(IV-55)}$$

Result

Differential correcting formulas in the case of ray tracing for all kinds of rays involving skew rays and imaginary rays, are derived.

Besides, correcting formulas of Seidel's aberrations which can be computed by using only the values obtained from the calculation for a paraxial ray, are derived.

It is shown that the partial differential coefficients required for this calculation can be computed directly from the differential calculation mentioned in Part III.

Therefore, the differential calculation given in this treatise becomes most effective when applied to the calculation of aberration and that of differential correction.

Optical Properties of Ebert Spectrograph

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Abstract

Focal surface and reciprocal linear dispersion of Ebert spectrograph are described. The focal surface varies with the position of the plane grating. When the distance between the spherical mirror and the plane grating is $R(1-\frac{1}{\sqrt{3}})$, where R is the radius of the mirror, the focal surface is a plane. When the distance is $\frac{1}{4}R$, the direction of the dispersion is perpendicular to the grating grooves. The reciprocal linear dispersion varies with the inclination of the grating and also by the position on the focal surface.

1. Introduction

Optical properties of Ebert monochromator when it is used in conjunction with a spectrograph are discussed. An attempt was made to clarify by calculation the relation of the position of grating to the curvature of focal curve and the dependency of dispersion on diffraction angle.

2. Tangential and sagittal focal curves

In Fig. 1, O is the center of the spherical mirror and G is an arbitrary point

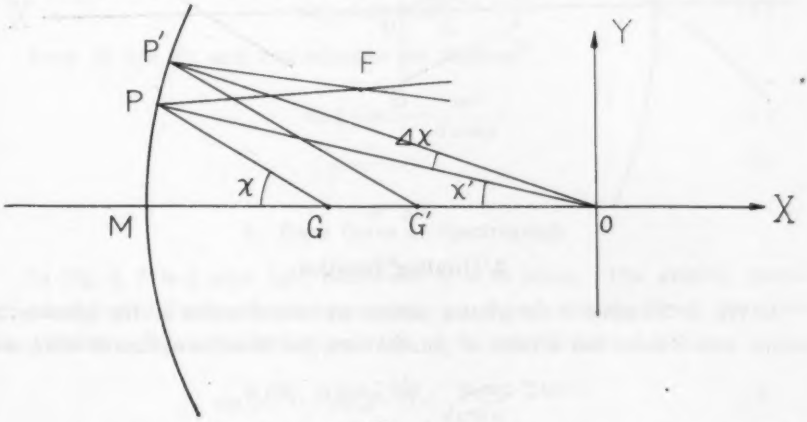


Fig. 1.

on the line MO. The origin of the rectangular coordinate is at O. X-axis coincides with MO and Y-axis is perpendicular to it. Let $GP, G'P'$ be two parallel rays in XY-plane intersecting at F after reflection at the mirror. Then the coordinates of $F(X, Y)$ are

$$X=-\frac{R}{2}+\frac{R}{4}\left(1-\frac{3g^2}{R^2}\right)\sin^2\chi+\cdots\cdots \tag{1}$$

$$Y=\frac{R}{2}\sin\chi+\left(\frac{3}{4}+\frac{g}{R}\right)\frac{g^2}{R}\sin^3\chi+\cdots\cdots \tag{2}$$

where R is the radius of the mirror, g is the X-coordinate of G, and χ is the angle PGM in the figure.

In Fig. 2, $G'P'$ and $G''P''$ are two rays, both parallel to PG, lying in a plane perpendicular to XY-plane. After reflection they intersect at F' which is in XY-plane. The coordinates of $F(X', Y')$ are

$$X'=-\frac{R}{2}+\frac{R}{4}\left(1-\frac{g^2}{R^2}\right)\sin^2\chi+\cdots\cdots \tag{3}$$

$$Y'=\frac{R}{2}\sin\chi+\frac{1}{4}\frac{g^2}{R}\sin^3\chi+\cdots\cdots. \tag{4}$$

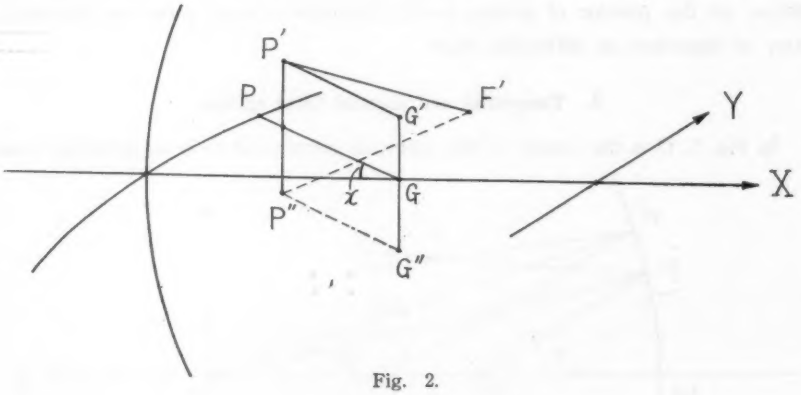


Fig. 2.

3. Grating Equation

In Fig. 3, YZ-plane is the grating surface and the direction of the grooves is parallel with Z-axis. Let a beam of parallel rays, the direction cosines of which are

$$-\cos\zeta\cos\mu,\quad -\sin\zeta\cos\mu,\quad \sin\mu$$

be incident on the grating and let direction cosines of the diffracted rays be

$$\cos\eta\cos\nu,\quad \sin\eta\cos\nu,\quad \sin\nu.$$

The optical path difference between two rays diffracted from two adjacent grooves must be a multiple of the wavelength. Hence, the difference is

$$d \sin\zeta \cos\mu + d \sin\eta \cos\nu = m\lambda \qquad (5)$$

where d is the grating constant.

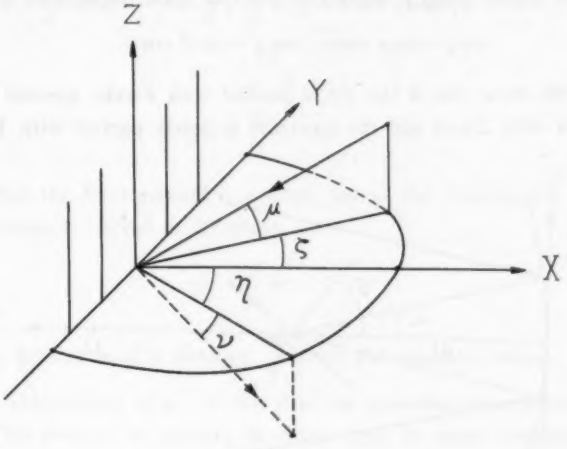


Fig. 3.

The optical difference of two rays diffracted from the same groove must be zero. Hence,

$$\sin\mu + \sin\nu = 0 \qquad (6)$$

From (5) and (6), next two relations are obtained

$$\sin\zeta + \sin\eta = \frac{m\lambda}{d\cos\varphi} \qquad (7)$$

$$\mu = -\nu . \qquad (8)$$

4. Focal Curve of Spectrograph

In Fig. 4, P is a point light source and Q is its image. The grating grooves are parallel with Z-axis. Among the angles α, φ, β and ψ in the figure there exist two relations. When the inclination angle of the grating normal is θ,

$$\sin(\alpha-\theta) + \sin(\beta-\theta) = \frac{m\lambda}{d\cos\varphi} \qquad (9)$$

$$\varphi = -\psi \qquad (10)$$

If $\alpha+\beta>2\theta$, m is positive and if $\alpha+\beta<2\theta$ m is negative.

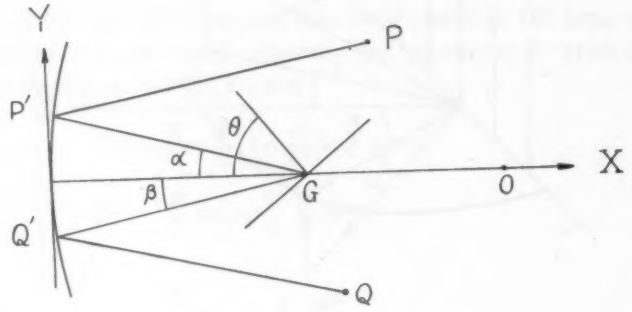
Tnagents of angles ε and ε' which the planes OGP and OGQ make respectively with XY-plane are

$$\tan\varepsilon=\frac{\tan\varphi}{\sin\alpha} \qquad \tan\varepsilon'=\frac{\tan\psi}{\sin\beta}\ . \tag{11}$$

Cosines of angles χ and χ' which GP' and GQ' make respectively with X-axis are

$$\cos\chi=\cos\alpha\ \cos\varphi \qquad \cos\chi'=\cos\beta\ \cos\psi\ . \tag{12}$$

(9) and (10) show that if the slit is parallel with Z-axis, spectral lines appear nearly parallel with Z-axis and the spectrum is nearly parallel with Y-axis.



top view

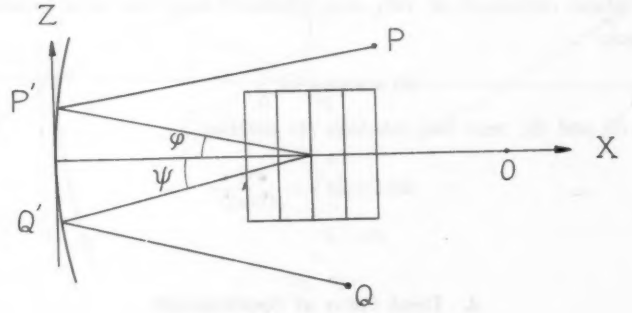


Fig. 4. side view

In the following, the position of the slit is assumed to be near X-axis and its direction coincide with that of the Z-axis. In such a case the rays which are incident upon the garting can be regarded as parallel rays and so can be also the diffracted rays. Taking into consideration that the direction of the slit is parallel with that of Z-axis and also that the angle ψ of the diffracted rays is small, we obtain from (1), (2), (11) and (12) the coordinates $Q(X, Y, Z)$ of the image point

$$X=-\frac{R}{2}+\frac{R}{4}\left(1-\frac{3g^2}{R^2}\right)(\sin^2\beta+\sin^2\varphi) \tag{13}$$

$$Y=\frac{R}{2}\sin\beta\cos\varphi+\left(\frac{3}{4}+\frac{g}{R}\right)\frac{g^2}{R}(\sin^2\beta+\sin^2\varphi)\sin\beta \tag{14}$$

$$Z=-\frac{R}{2}\sin\varphi-\left(\frac{3}{4}+\frac{g}{R}\right)\frac{g^2}{R}(\sin^2\beta+\sin^2\varphi)\sin\varphi\,. \tag{15}$$

If $$g=-\frac{R}{\sqrt{3}}$$

then $$X=-\frac{R}{2}\,.$$

This means that the focal surface is a plane, but as the Z-coordinate is a function of β, the spectrum is curved in XZ-plane.

If $$g=-\frac{3}{4}R$$

that is, if the grating is at a distance $\frac{R}{4}$ from the spherical mirror, the Z-coordinate becomes independent of β. In this case the spectrum is perfectly parallel with XY-plane. This position of grating is being used in some commercial spectrographs.[1]

5. Dispersion

When g is equal to $-\frac{3}{4}R$

$$X=-\frac{1}{2}R-\frac{11}{64}R(\sin^2\beta+\sin^2\varphi)$$

$$Y=\frac{1}{2}R\sin\beta\cos\varphi$$

$$Z=-\frac{1}{2}R\sin\varphi$$

therefore

$$dS=(dX^2+dY^2)^{\frac{1}{2}}=\left|\left(\frac{11}{32}\sin\beta\right)^2+\frac{1}{4}\cos^2\varphi\right|^{\frac{1}{2}}R\cos\beta\,d\beta \tag{16}$$

From (9) and (16) the reciprocal linear dispersion is derived as

$$\frac{d\lambda}{dS}=\frac{2d}{mR}\left\{\cos\theta+\sin\theta\cdot\frac{2Y}{R}\left(1+2\frac{Z^2}{R^2}\right)\right\}.$$

The reciprocal linear dispersion decreases as θ increases.

1) R. F. Jarrel J.O.S.A. **45** (1955) 259.

As the distance of the slit from XY-plane increases, the dispersion varies.

The wavelength λ_0 of the rays that focus on the line $Y=0$ of the focal surface is

$$\lambda_0=\frac{2d}{m}\cos\frac{\alpha}{2}\sin\left(\frac{\alpha}{2}-\theta\right)\left(1-\frac{1}{2}\sin^2\varphi\right). \tag{17}$$

When φ is very small, λ_0 may be considered constant but, with the increase of φ, spectral lines begin to till against XZ-plane.

Acknowledgement

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